

Distributed dual gradient methods and error bound conditions

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Abstract In this paper¹ we propose distributed dual gradient algorithms for linearly constrained separable convex problems and analyze their rate of convergence under different assumptions. Under the strong convexity assumption on the primal objective function we propose two distributed dual fast gradient schemes for which we prove sublinear rate of convergence for dual suboptimality but also primal suboptimality and feasibility violation for an average primal sequence or for the last generated primal iterate. Under the additional assumption of Lipschitz continuity of the gradient of the primal objective function we prove a global error bound type property for the dual problem and then we analyze a dual gradient scheme for which we derive global linear rate of convergence for both dual and primal suboptimality and primal feasibility violation. We also provide numerical simulations on optimal power flow problems.

1 Introduction

Nowadays, many engineering applications which appear in the context of communications networks or networked systems can be posed as linearly constrained separable convex problems. Several important applications that can be modeled in this framework, the network utility maximization (NUM) problem [1], the optimal power flow (DC-OPF) problem for a power system [28] and distributed model predictive control (MPC) problem for networked systems [10], have attracted great attention lately. Due to the large dimension and the separable structure of these problems, distributed optimization methods have become an appropriate tool for solving such problems.

The standard approach to distributed optimization in networks is to use decomposition. Decomposition methods represent a powerful tool for solving these type of problems due to their ability of dividing the original large scale problem into

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smaller subproblems which are coordinated by a master problem. Decomposition methods can be divided in two main classes: primal and dual decomposition. While in the primal decomposition methods the optimization problem is solved using the original formulation and variables, in dual decomposition the constraints are moved into the cost using the Lagrange multipliers and the dual problem is solved. In many applications, such as (NUM), (DC-OPF) and (MPC) problems, when the constraints set is complicated (i.e. the projection on this set is hard to compute) dual decomposition becomes more effective since a primal approach will require at each iteration a projection onto the feasible set, operation that is numerically very expensive.

First order methods for solving dual problems have been extensively studied in the literature. Subgradient methods based on averaging, that produce primal solutions in the limit, can be found e.g. in [4, 6, 24]. Despite widespread use of the (sub)gradient methods for solving dual problems, there are some aspects that have not been fully studied. In particular, in practical applications, the main interest is in finding an approximate primal solution that is near-feasible and near-optimal. Moreover, we need to characterize the convergence rate for the approximate primal solution. Finally, we are interested in providing distributed schemes, i.e. methods based on distributed computations. These represent the main issues that we pursue in this paper.

Convergence rate analysis for the dual subgradient method has been studied e.g. in [2, 15], where estimates of order $\mathcal{O}(1/\sqrt{k})$ for suboptimality and feasibility violation of an average primal sequence are provided, with k denoting the iteration counter. In [13] the authors propose a dual fast gradient algorithm based on a smoothing technique and prove rate of convergence of order $\mathcal{O}(\frac{1}{k})$ for primal suboptimality and feasibility violation for an average primal sequence. Also, in [10] the authors propose inexact dual (fast) gradient algorithms for which estimates of order $\mathcal{O}(\frac{1}{k})$ ($\mathcal{O}(\frac{1}{k^2})$) in an average primal sequence are provided for both primal and dual suboptimality and primal feasibility violation. For the special case of QPs problems, dual gradient algorithm were also analyzed in [5, 19, 20]. From our knowledge first result on the linear convergence of dual gradient method was provided in [8]. However, the authors in [8] were able to show linear convergence only locally. Finally, very few results were known in the literature on distributed implementations of dual gradient type methods since most of the papers enumerated above require a centralized step size. Recently, the authors in [1] propose a distributed dual fast gradient algorithm where the step size is chosen distributively and provide estimates of order $\mathcal{O}(\frac{1}{k})$ for primal suboptimality and feasibility violation in the last primal iterate. All of these limitation motivates our work here. In this paper we propose distributed versions of dual first order methods generating approximate primal feasible and primal optimal solutions but with great improvement on the convergence rate w.r.t. the existing results from the literature. In particular, under the strong convexity assumption on the primal objective function we derive a distributed version of the dual fast gradient algorithm presented in [10] for which we are able to provide estimates of order $\mathcal{O}(\frac{1}{k^2})$ on primal suboptimality and feasibility violation for an average primal sequence. In comparison with the algorithm proposed in [10] we do not require a centralized step size and thus we derive a distributed implementation of the algorithm. Also, the estimates on primal suboptimality and feasibility violation for our distributed algorithm are with an order of magnitude better than the ones of algorithm given in [1]. We also

propose a hybrid dual fast gradient algorithm which allows us to provide estimates of order $\mathcal{O}\left(\frac{1}{k\sqrt{k}}\right)$ on primal suboptimality and feasibility violation in the last primal iterate. Note that also in this case the iteration complexity of our method is better than of the method given in [1]. Under the additionally Lipschitz continuity assumption on the gradient of primal objective function, which is often satisfied in practical applications (e.g. (NUM) and (MPC) problems), we prove that the corresponding dual problem satisfies a certain error bound property [8]. In order to prove such a property we extend the approach developed in [8, 26] to the case when the constraints set is an unbounded polyhedron. In these settings we analyze the convergence behavior of a distributed dual gradient algorithm for which we are able to provide for the first time *global* linear convergence rate on primal suboptimality and infeasibility for the last primal iterate, as opposed to the results in [8] where only *local* linear convergence was derived for such an algorithm. We also show that the theoretical estimates on the convergence rate depend on a natural and easily computable measure of separability of the problem.

Contribution. In summary, the contributions of this paper include:

- (i) We propose and analyze novel dual gradient type algorithms having distributed implementations and fast rate of convergence that generate approximate primal solutions for separable (smooth) convex problems with linear constraints.
- (ii) For these distributed algorithms we derive estimates on primal suboptimality and infeasibility in an average/last sequence: a dual fast gradient method with convergence rate $\mathcal{O}(1/k^2)$ in an average primal sequence; an hybrid dual fast gradient method with convergence rate $\mathcal{O}(1/k^{3/2})$ in the last primal iterate; a dual gradient method with linear convergence in the last primal iterate.
- (iii) Under strong convexity and Lipschitz continuity of the gradient of the primal objective function we prove an error bound type property for the dual problem which allows us to obtain global linear convergence for a distributed dual gradient method.

Paper Outline. In Section 2 we introduce our optimization model and discuss several practical applications which can be posed in this framework. In Sections 3 and 4 we propose two distributed dual fast gradient algorithms and provide sublinear estimates for both dual and primal suboptimality, but also for primal feasibility violation in an average primal sequence or in the last generated primal iterate. In Section 5 we show that under additional assumptions on the primal objective function the dual problem has some error bound property which allows us to prove global linear converge for a distributed dual gradient method. Finally, in Section 7 we provide extensive numerical simulations in order to certify our proposed theory.

Notations: We work in the space \mathbb{R}^n composed of column vectors. For $z, y \in \mathbb{R}^n$ we denote the standard Euclidean inner product $\langle z, y \rangle = \sum_{i=1}^n z_i y_i$, the Euclidean norm $\|z\| = \sqrt{\langle z, z \rangle}$ and the infinity norm $\|z\|_\infty = \sup_i |z_i|$. Also, w.r.t. to the Euclidean norm $\|\cdot\|$ we denote the projection onto the non-negative orthant \mathbb{R}_+^n by $[z]_+$ and the projection onto the convex set D by $[z]_D$. For a positive definite matrix W we denote the weighted norm of a vector z by $\|z\|_W^2 = z^T W z$ and the projection of the vector z onto a convex set D w.r.t. to norm $\|\cdot\|_W$ by $[z]_D^W$. For a (block) matrix A we define by A_i its i th (block) column. We denote by I_q the identity matrix in $\mathbb{R}^{q \times q}$ and by $0_{p,q}$ the matrix from $\mathbb{R}^{p \times q}$ with all entries zero.

2 Problem formulation

We consider the following linearly constrained separable convex optimization problem:

$$\begin{aligned} f^* = \min_{z_i \in \mathbb{R}^{n_i}} f(z) \quad & \left(= \sum_{i=1}^M f_i(z_i) \right) \\ \text{s.t.: } & Az = b, \quad Cz \leq c, \end{aligned} \quad (1)$$

where f_i are convex functions, $z = [z_1^T \dots z_M^T]^T$, $A \in \mathbb{R}^{p \times n}$, $C \in \mathbb{R}^{q \times n}$, $b \in \mathbb{R}^p$ and $c \in \mathbb{R}^q$. To our optimization problem (1) we associate a communication bipartite graph $\mathcal{G} = (V_1, V_2, E)$, where $V_1 = \{1, \dots, M\}$, $V_2 = \{1, \dots, \bar{M}\}$ and $E \in \{0, 1\}^{(\bar{M}) \times M}$ is an incidence matrix. We also introduce the index sets $\mathcal{N}_i = \{j \in V_2 : E_{ij} \neq 0\}$ for all $i \in V_1$ and $\bar{\mathcal{N}}_j = \{i \in V_1 : E_{ij} \neq 0\}$ for all $j \in V_2$ which describe the local information flow in the graph. Note that the cardinality of the sets \mathcal{N}_i and $\bar{\mathcal{N}}_j$ can be viewed as a measure for the degree of separability of problem (1). Therefore, the local information structure imposed by the graph \mathcal{G} should be considered as part of the problem formulation. We assume that A and C are block matrices with the blocks $A_{ij} \in \mathbb{R}^{p_i \times n_i}$ and $C_{ij} \in \mathbb{R}^{q_i \times n_i}$, where $\sum_{i=1}^M n_i = n$, $\sum_{i=1}^M p_i = p$ and $\sum_{i=1}^M q_i = q$. We also assume that if $E_{ij} = 0$, then both blocks A_{ij} and C_{ij} are zero. In these settings we allow a block A_{ij} or C_{ij} to be zero even if $E_{ij} = 1$.

Further, we make the following assumption on the optimization problem (1):

- Assumption 1** (a) *The functions f_i are σ_i -strongly convex w.r.t. Euclidean norm $\|\cdot\|$ [16].*
 (b) *The feasible set of problem (1) is nonempty and there exists \bar{z} such that $A\bar{z} = b$ and $C\bar{z} < c$.*

Note that if Assumption 1 (a) does not hold, we can apply smoothing techniques by adding a regularization term to the function f_i in order to obtain a strongly convex approximation of it (see e.g. [13] for more details). Assumption 1 (b) implies that strong duality holds for optimization problem (1) and the set of optimal Lagrange multipliers is bounded [3, 9]. In particular, we have:

$$f^* = \max_{\nu \in \mathbb{R}^p, \mu \in \mathbb{R}_+^q} d(\nu, \mu), \quad (2)$$

where $d(\nu, \mu)$ denote the dual function of (1):

$$d(\nu, \mu) = \min_{z \in \mathbb{R}^n} \mathcal{L}(z, \nu, \mu), \quad (3)$$

with the Lagrangian function $\mathcal{L}(z, \nu, \mu) = f(z) + \langle \nu, Az - b \rangle + \langle \mu, Cz - c \rangle$. For simplicity of the exposition we introduce further the following notations:

$$G = \begin{bmatrix} A \\ C \end{bmatrix} \quad \text{and} \quad g = \begin{bmatrix} b \\ c \end{bmatrix}. \quad (4)$$

Since f_i are strongly convex functions, then f is also strongly convex w.r.t. Euclidean norm $\|\cdot\|$ with convexity parameter $\sigma_f = \min_{i=1, \dots, M} \sigma_i$. Further, the dual function d is differentiable and its gradient is given by the following expression [10]:

$$\nabla d(\nu, \mu) = Gz(\nu, \mu) - g,$$

where $z(\nu, \mu)$ denotes the unique optimal solution of the inner problem (3), i.e.:

$$z(\nu, \mu) = \arg \min_{z \in \mathbb{R}^n} \mathcal{L}(z, \nu, \mu). \quad (5)$$

Moreover, the gradient ∇d of the dual function is Lipschitz continuous w.r.t. Euclidean norm $\|\cdot\|$, with constant [10]:

$$L_d = \frac{\|G\|^2}{\sigma_f}.$$

If we denote by $\nu_{\mathcal{N}_i} = [\nu_j]_{j \in \mathcal{N}_i}$ and by $\mu_{\mathcal{N}_i} = [\mu_j]_{j \in \mathcal{N}_i}$ we can observe that the dual function can be written in the following separable form:

$$d(\nu, \mu) = \sum_{i=1}^M d_i(\nu_{\mathcal{N}_i}, \mu_{\mathcal{N}_i}) - \langle \nu, b \rangle - \langle \mu, c \rangle,$$

with

$$\begin{aligned} d_i(\nu_{\mathcal{N}_i}, \mu_{\mathcal{N}_i}) &= \min_{z_i \in \mathbb{R}^{n_i}} f_i(z_i) + \langle \nu, A_i z_i \rangle + \langle \mu, C_i z_i \rangle \\ &= \min_{z_i \in \mathbb{R}^{n_i}} f_i(z_i) + \sum_{j \in \mathcal{N}_i} \langle A_{ji}^T \nu_j + C_{ji}^T \mu_j, z_i \rangle. \end{aligned} \quad (6)$$

In these settings, we have that the gradient ∇d_i is given by:

$$\nabla d_i(\nu_{\mathcal{N}_i}, \mu_{\mathcal{N}_i}) = \begin{bmatrix} [A_{ji}]_{j \in \mathcal{N}_i} \\ [C_{ji}]_{j \in \mathcal{N}_i} \end{bmatrix} z_i(\nu_{\mathcal{N}_i}, \mu_{\mathcal{N}_i}),$$

where $z_i(\nu_{\mathcal{N}_i}, \mu_{\mathcal{N}_i})$ denotes the unique optimal solution in (6). Note that ∇d_i is Lipschitz continuous w.r.t. Euclidean norm $\|\cdot\|$, with constant:

$$L_{d_i} = \frac{\left\| \begin{bmatrix} [A_{ji}]_{j \in \mathcal{N}_i} \\ [C_{ji}]_{j \in \mathcal{N}_i} \end{bmatrix} \right\|^2}{\sigma_i}. \quad (7)$$

For simplicity of the exposition we will consider further the notations:

$$\lambda = \begin{bmatrix} \nu^T & \mu^T \end{bmatrix}^T \quad \text{and} \quad \lambda_j = \begin{bmatrix} \nu_j^T & \mu_j^T \end{bmatrix}^T \quad \forall j \in V_2,$$

and we will also denote the effective domain of the dual function by $\mathbb{D} = \mathbb{R}^p \times \mathbb{R}_+^q$. The following result, which is a distributed version of descent lemma is central in our derivations of distributed algorithms and in our proofs for the convergence rate for them. Note that a similar result for the case of inequality constraints can be also found in [1].

Lemma 1 *Let Assumption 1 (a) hold. Then, the following inequality is valid:*

$$d(\lambda) \geq d(\bar{\lambda}) + \langle \nabla d(\bar{\lambda}), \lambda - \bar{\lambda} \rangle - \frac{1}{2} \|\lambda - \bar{\lambda}\|_W^2 \quad \forall \lambda, \bar{\lambda} \in \mathbb{D}, \quad (8)$$

where the matrix $W = \text{diag}(W_\nu, W_\mu)$ with $W_\nu = \text{diag}\left(\sum_{i \in \tilde{\mathcal{N}}_j} L_{d_i} I_{p_j}; j \in V_2\right)$ and $W_\mu = \text{diag}\left(\sum_{i \in \tilde{\mathcal{N}}_j} L_{d_i} I_{q_j}; j \in V_2\right)$.

Proof Let us first denote by $\lambda_{\mathcal{N}_i} = [\nu_{\mathcal{N}_i}^T \mu_{\mathcal{N}_i}^T]^T$. Using now the continuous Lipschitz gradient property of d_i we can write for each $i = 1, \dots, M$:

$$d_i(\lambda_{\mathcal{N}_i}) \geq d_i(\bar{\lambda}_{\mathcal{N}_i}) + \langle \nabla d_i(\bar{\lambda}_{\mathcal{N}_i}), \lambda_{\mathcal{N}_i} - \bar{\lambda}_{\mathcal{N}_i} \rangle - \frac{L_{d_i}}{2} \|\lambda_{\mathcal{N}_i} - \bar{\lambda}_{\mathcal{N}_i}\|^2.$$

Summing up these inequalities for all $i = 1, \dots, M$ and adding $\langle \lambda, [b^T c^T]^T \rangle$ to both sides of the obtained inequality we obtain:

$$d(\lambda) \geq d(\bar{\lambda}) + \langle \nabla d(\bar{\lambda}), \lambda - \bar{\lambda} \rangle - \sum_{i=1}^M \frac{L_{d_i}}{2} \|\lambda_{\mathcal{N}_i} - \bar{\lambda}_{\mathcal{N}_i}\|^2. \quad (9)$$

Using now the definitions of $\lambda_{\mathcal{N}_i}$ and W we can write:

$$\sum_{i=1}^M \frac{L_{d_i}}{2} \|\lambda_{\mathcal{N}_i} - \bar{\lambda}_{\mathcal{N}_i}\|^2 = \sum_{j=1}^{\bar{M}} \sum_{i \in \mathcal{N}_j} \frac{L_{d_i}}{2} \|\lambda_j - \bar{\lambda}_j\|^2 = \frac{1}{2} \|\lambda - \bar{\lambda}\|_W.$$

Introducing this result into the previous inequality we conclude the statement. \square

The following result, which is an extension of Lemma 2.2 in [1] to the case when both equality and inequality constraints are present, will be useful for characterizing the distance between a primal estimate and the primal optimal solution z^* of our optimization problem (1).

Lemma 2 *Let Assumption 1 hold. Then, the following relation is valid:*

$$\frac{\sigma_f}{2} \|z(\lambda) - z^*\|^2 \leq f^* - d(\lambda) \quad \forall \lambda \in \mathbb{D}, \quad (10)$$

where $z(\lambda) = \arg \min_{z \in \mathbb{R}^n} \mathcal{L}(z, \lambda)$.

Proof Since f is σ_f -strongly convex it follows that $\mathcal{L}(z, \lambda)$ is also σ_f -strongly convex in the variable z which together with the definition of $d(\lambda) = f(z(\lambda)) + \langle \lambda, Gz(\lambda) - g \rangle$ and $\nabla d(\lambda) = Gz(\lambda) - g$ and the fact that $\langle \lambda, \nabla d(\lambda^*) \rangle \leq 0$ gives:

$$\begin{aligned} \frac{\sigma_f}{2} \|z(\lambda) - z^*\| &\leq \mathcal{L}(z^*, \lambda) - \mathcal{L}(z(\lambda), \lambda) \\ &= f(z^*) + \langle \lambda, \nabla d(\lambda^*) \rangle - f(z(\lambda)) - \langle \lambda, \nabla d(\lambda) \rangle \leq f^* - d(\lambda). \end{aligned}$$

\square

We denote by Λ^* the set of optimal solutions of dual problem (2). According to Gauvin's theorem [3], if Assumption 1 holds for our original problem (1), then Λ^* is nonempty and bounded. Since the set of optimal Lagrange multipliers is bounded, for any $\lambda^0 \in \mathbb{R}^{p+q}$ we can define the following finite quantity:

$$\mathcal{R}(\lambda^0) = \max_{\lambda^* \in \Lambda^*} \|\lambda^* - \lambda^0\|_W. \quad (11)$$

In this paper we propose different distributed dual first order methods for which we are interested in deriving estimates for both dual and primal suboptimality and also for primal feasibility violation, i.e. finding a primal-dual pair $(\hat{z}, \hat{\lambda})$ such that:

$$\begin{aligned} \|G\hat{z} - g\|_{\mathbb{D}} &\leq \mathcal{O}(\epsilon), \quad \|\hat{z} - z^*\|^2 \leq \mathcal{O}(\epsilon), \\ -\mathcal{O}(\epsilon) &\leq f(\hat{z}) - f^* \leq \mathcal{O}(\epsilon) \quad \text{and} \quad f^* - d(\hat{\lambda}) \leq \mathcal{O}(\epsilon), \end{aligned} \quad (12)$$

where ϵ is a given accuracy.

2.1 Motivation

Many engineering applications from networks can be posed as linearly constrained separable convex optimization problems of type (1). We will discuss further three such applications, namely network utility maximization (NUM) problem, optimal power flow (DC-OPF) problem for a power system and distributed model predictive control (MPC) problem for networked systems.

2.1.1 Network utility optimization

We consider a network characterized by a bipartite graph $\mathcal{G} = (V_1, V_2, E)$, with $V_1 = \{1, \dots, M\}$ a set of sources, $V_2 = \{1, \dots, \bar{M}\}$ a set of capacitated links, each link j having capacity $\bar{c}_j > 0$, and E its incidence matrix. In these settings, \mathcal{N}_i represents the set of links which are used by the source i , while $\bar{\mathcal{N}}_j$ is the set of sources which share the link j . Also, we attach to each source i a strongly convex decreasing utility function $f_i(z_i)$, where $z_i \in \mathbb{R}$ denotes the rate at which the source sends its data. In these settings, the goal of the network utility problem is to find the optimal rates at which the total utility function is maximized. Introducing the notation $z = [z_1 \dots z_M]^T$, the network utility maximization problem can be posed as the following convex optimization problem:

$$\begin{aligned} f^* = \min_{z_i \in \mathbb{R}} f(z) \quad & \left(= \sum_{i=1}^M f_i(z_i) \right) \\ \text{s.t.:} \quad & \sum_{i \in \mathcal{N}_j} z_i \leq \bar{c}_j \quad \forall j \in V_2, \quad z_i \in Z_i = [0, R_i] \quad \forall i \in V_1. \end{aligned} \quad (13)$$

By stacking together all the local and coupling constraints, we can observe that problem (13) can be written in the form of problem (1), where the equality constraints are absent. Well known applications are the NUM problem [1] and dynamic network utility maximization (DNUM) with end-to-end delays [25].

2.1.2 DC Optimal power flow

Let us discuss the active optimal power flow (DC-OPF) problem for a power system [28]. We consider a power system whose structure is characterized by a directed bipartite graph $\mathcal{G} = (V_1, V_2, E)$, where $V_1 = \{i \mid i = 1, \dots, M\}$ denotes the set of buses, $V_2 = \{l = (i, j) \mid i, j \in V_1, l = 1, \dots, \bar{M}\} \subseteq V_1 \times V_1$ represent the sets of transmission lines (branches) between two buses and the matrix E denotes its incidence matrix. In these settings we have:

$$\mathcal{N}_i = \{l \in V_2 \mid E_{li} \neq 0\} = \{l \in V_2 \mid \exists j \in V_1 \text{ s.t. } (i, j) \vee (j, i) = l\}$$

which denotes the set of all transmission lines from or to bus i and

$$\bar{\mathcal{N}}_l = \{i \in V_1 \mid E_{li} \neq 0\} = \{i, j \in V_1 \mid (i, j) \vee (j, i) = l\}$$

which denotes the set comprised of buses i and j which define the branch l . We also introduce:

$$\mathcal{S}_i = \bigcup_{l \in V_2} \{j \in V_1 \mid E_{li} \neq 0 \wedge E_{lj} \neq 0\} = \{j \in V_1 \mid \exists l \in V_2 \text{ s.t. } (i, j) \vee (j, i) = l\}$$

which denotes the sets of all buses directly linked with bus i . It is straightforward to notice that the set \mathcal{S}_i can be obtained from the sets \mathcal{N}_i and $\bar{\mathcal{N}}_i$.

We define further the diagonal matrix $R \in \mathbb{R}^{M \times M}$, whose diagonal elements R_{ll} represent the reactance of the l th transmission line between two busses i and $j \in V_1$. For each bus i we denote by

$$\theta_i \in \Theta_i = [\underline{\theta}_i, \bar{\theta}_i]$$

the phase angle of the voltage and by

$$P_i^g \in \mathcal{P}_i = [\underline{P}_i^g, \bar{P}_i^g]$$

the generated power if the bus i is directly connected to a generator. Under this model, the active power flow from a bus i to a bus j is given by:

$$F_l = \frac{1}{R_{ll}} (\theta_i - \theta_j), \quad (14)$$

where $l = (i, j)$ and we recall that R_{ll} represent the reactance of the transmission line connecting buses i and j . We impose lower and upper line flows limits $\underline{F} = [\underline{F}_1 \cdots \underline{F}_M]^T$ and $\bar{F} = [\bar{F}_1 \cdots \bar{F}_M]^T$, respectively. We also assume that each bus i is characterized by a local load P_i^d and we denote by $P^d = [P_1^d \cdots P_M^d]^T$ the overall vector of loads. We introduce further the notations:

$$\theta = [\theta_1 \cdots \theta_M]^T \text{ and } P^g = [P_1^g \cdots P_{M_g}^g]^T,$$

where M_g denotes the number of generators. We also define the matrix $A^g \in [0, 1]^{M \times M_g}$ having $A_{ij}^g = 1$ if P_j^g is directly linked with the bus i and the rest of its entry equal to zero. Note that if we consider that each bus i is directly coupled with a generator unit $A^g = I_M$. Using these notations, the DC nodal power balance can be written in the following form [28]:

$$E^T R E \theta = A^g P^g - P^d, \quad (15)$$

where the matrix $E^T R E$ denotes the weighted Laplacian and its entries have the following expressions:

$$[E^T R E]_{ij} = \begin{cases} \sum_{s \in \mathcal{S}_i} R_{ls}, & l = (i, s) \vee (s, i) \text{ if } i = j \\ -R_{ll}, & l = (i, j) \vee (j, i) \text{ if } i \neq j \\ 0 & \text{otherwise.} \end{cases}$$

We can observe that the structure of the Laplacian matrix is given by the structure of the incidence matrix E through the sets \mathcal{S}_i , which, at its turn depend on the sets \mathcal{S}_i and \mathcal{N}_l for all $i \in V_1$ and $l \in V_2$. Using further the relation between the the power flow and the phase angle of the voltages, we can write the lower and upper limits imposed on the line flows in the following matrix form:

$$\underline{F} \leq R E \theta \leq \bar{F}. \quad (16)$$

We also define reference values θ_i^{ref} for the phase angle of the voltage of each bus and $P_i^{g,\text{ref}}$ for the generated powers of each generator. Further, for each bus i we define a local decision variable z_i as follows:

$$z_i = \begin{cases} \begin{bmatrix} \theta_i \\ P_i^g \end{bmatrix} & \text{if the bus } i \text{ is connected to a generator} \\ \theta_i & \text{otherwise} \end{cases}$$

and the corresponding reference values z_i^{ref} .

In comparison with the approach made in [28], where the authors consider the lower and upper limits of the form $\theta_i^{\text{ref}} \leq \theta_i \leq \theta_i^{\text{ref}}$, in our approach we do not impose such constraints but use instead a weighted quadratic cost, which, depending on the value of the parameter q_i , requires the solution to be close to the reference value θ_i^{ref} . The main motivation behind this approach consist in the fact that constraints of this form usually induce numerical problems due to the fact that the optimization problem which has to be solved is badly conditioned (for example, the Slater constraint qualification does not hold in this case). Therefore, for each bus i directly connected to a generator unit we impose a local cost of the form:

$$f_i(z_i) = 0.5 \|z_i - z_i^{\text{ref}}\|_{Q_i}^2 - \gamma_i \log(\beta_i + P_i^g), \quad (17)$$

where the diagonal matrix $Q_i = \begin{bmatrix} q_i & 0 \\ 0 & p_i \end{bmatrix} \in \mathbb{R}^{2 \times 2}$ and the positive scalar γ_i are used in order to weight the local cost. Also, the positive scalar β_i is used to avoid numerical instability when P_i^g is closed to 0. Also, in comparison with the existing approaches for (DC-OPF) problems we add to the classic quadratic term a weighted logarithmic term, which is used in many resource allocation problem (see e.g. [27]) in order to reduce the absolute risk aversion. The logarithmic utility function also exhibit diminishing returns with the rate of resources, in our case the generated power, that is, as rate increases the incremental utility grows by smaller amounts. For the buses that are not connected to a generator unit we impose a simple quadratic local cost of the form:

$$f_i(z_i) = 0.5 q_i \left(z_i - \theta_i^{\text{ref}} \right)^2, \quad (18)$$

where in this case q_i is a positive scalar. Note that for these choices the local costs f_i are strongly convex functions for both cases. In conclusion, the (DC-OPF) problem can be cast as the following large-scale separable convex optimization problem:

$$\begin{aligned} f^* = \min_{\theta_i \in \Theta_i, P_i^g \in \mathcal{P}_i} & \sum_{i_1} f_{i_1}(\theta_{i_1}) + \sum_{i_2} f_{i_2}(\theta_{i_2}, P_{i_2}^g) \\ \text{s.t.: } & E^T RE\theta - A^g P^g = -P^d, \quad \underline{F} \leq RE\theta \leq \overline{F}. \end{aligned} \quad (19)$$

2.1.3 Distributed MPC for networked systems

We consider a discrete-time networked system, modelled by a graph $\mathcal{G} = (V, E)$, for which the set $V = \{1, \dots, M\}$ represents the subsystems and the adjacency matrix

E indicates the dynamic couplings between these subsystems. The dynamics of the subsystems can be defined by the following linear state equations [13]:

$$x_i(t+1) = \sum_{j \in \mathcal{N}_i} \bar{A}_{ij} x_j(t) + \bar{B}_{ij} u_j(t) \quad \forall i \in V, \quad (20)$$

where $x_i(t) \in \mathbb{R}^{n_{x_i}}$ and $u_i(t) \in \mathbb{R}^{n_{u_i}}$ represent the state and the input of i th subsystem at time t , $\bar{A}_{ij} \in \mathbb{R}^{n_{x_i} \times n_{x_j}}$ and $\bar{B}_{ij} \in \mathbb{R}^{n_{x_i} \times n_{u_j}}$. Note that in these settings \mathcal{N}_i denotes the set of subsystems, including i , whose dynamics directly affect the dynamics of subsystem i and $\bar{\mathcal{N}}_i$ represents the set of subsystem, including i , whose dynamics are affected by the dynamics of subsystem i . We also impose local state and input constraints:

$$x_i(t) \in X_i, \quad u_i(t) \in U_i \quad \forall i \in V, \quad t \geq 0,$$

where $X_i \subseteq \mathbb{R}^{n_{x_i}}$ and $U_i \subseteq \mathbb{R}^{n_{u_i}}$ are polyhedral sets. For a prediction horizon of length N , we consider strongly convex stage and final costs for each subsystem i :

$$\sum_{t=0}^{N-1} \ell_i(x_i(t), u_i(t)) + \ell_i^f(x_i(N)),$$

where the final costs ℓ_i^f are chosen such that the control scheme ensures stability [10, 13, 23]. The centralized MPC problem for the networked system (20), for a given initial state $x = [x_1^T \cdots x_M^T]^T$ can be posed as the following convex optimization problem:

$$\begin{aligned} \min_{x_i(t), u_i(t)} \quad & \sum_{i=1}^M \sum_{t=0}^{N-1} \ell_i(x_i(t), u_i(t)) + \ell_i^f(x_i(N)) \\ \text{s.t.} \quad & x_i(t+1) = \sum_{j \in \mathcal{N}^i} \bar{A}_{ij} x_j(t) + \bar{B}_{ij} u_j(t), \quad x_i(0) = x_i, \\ & x_i(t) \in X_i, \quad u_i(t) \in U_i, \quad x_i(N) \in X_i^f \quad \forall i \in V, \quad \forall t, \end{aligned} \quad (21)$$

where X_i^f are terminal sets chosen under some appropriate conditions to ensure stability of the MPC scheme (see e.g. [10, 13, 23]). For the state and input trajectory of subsystem i and the overall state and input trajectory we use the notations:

$$\begin{aligned} z_i &= \left[u_i(0)^T x_i(1)^T \cdots u_i(N-1)^T x_i(N)^T \right]^T, \\ z &= \left[z_1^T \cdots z_M^T \right]^T, \end{aligned}$$

and for the total local cost over the prediction horizon and local constraints of each subsystem we introduce:

$$\begin{aligned} f_i(z_i) &= \sum_{t=0}^{N-1} \ell_i(x_i(t), u_i(t)) + \ell_i^f(x_i(N)), \\ Z_i &= \left(\prod_{i=1}^{N-1} U_i \times X_i \right) \times U_i \times X_i^f. \end{aligned}$$

In these settings, the optimization problem (21) can be written equivalently as the structured optimization problem (1) where $n_i = N(n_{u_i} + n_{x_i})$, the equality constraints $Az = b$ are obtained by stacking all the dynamics (20) together, while the inequality constraints $Cz \leq c$ are obtained by writing the local constraints $z_i \in Z_i$ in compact form. Note also that for the matrix A , each block $A_{ij} = 0$ whenever $E_{ij} = 0$ and C is a block diagonal matrix.

In the following sections we will propose and analyze dual distributed (fast) gradient methods for solving the dual problem (2) which exploit the separability of the dual function and allow us to recover a suboptimal and nearly feasible solution for our original problem (1).

3 Distributed dual fast gradient algorithm (DFG)

In this section we propose a distributed dual fast (also called accelerated) gradient scheme (**DFG**) for solving the dual problem (2). A similar algorithm was proposed by Nesterov in [17] and applied further in [13] for solving dual problems. A similar version of the algorithm was also proposed in [10] for the case when the dual updates use inexact information and the step size is a fixed scalar. The scheme defines two sequences $(\hat{\lambda}^k, \lambda^k)_{k \geq 0}$ for the dual variables:

Algorithm (DFG)

Initialization: $\lambda^0 = 0$. For $k \geq 0$ compute:

1. $z^k = \arg \min_{z \in \mathbb{R}^n} \mathcal{L}(z, \lambda^k)$
2. $\hat{\lambda}^k = [\lambda^k + W^{-1} \nabla d(\lambda^k)]_{\mathbb{D}}$
3. $\lambda^{k+1} = \frac{k+1}{k+3} \hat{\lambda}^k + \frac{2}{k+3} \left[W^{-1} \sum_{s=0}^k \frac{s+1}{2} \bar{\nabla} d(\lambda^s) \right]_{\mathbb{D}}.$

For simplicity of the exposition we restrict our analysis to the case $\lambda^0 = 0$. Note that the behavior of the Algorithm (**DFG**) remain unchanged if one use any initialization $\lambda^0 \in \mathbb{D}$ (see e.g. [10]). We can also observe that step 1 of the algorithm requires an exact solution of the inner optimization problem. In many practical applications such a solution is hard to be computed. Instead, one can compute an approximate solution, i.e. $\bar{z}^k \approx \arg \min_{z \in \mathbb{R}^n} \mathcal{L}(z, \lambda^k)$, which satisfies a certain inner accuracy (see [10] for a detailed discussion). The main difference between our Algorithm (**DFG**) and the algorithms proposed in [10, 13, 17] consists in the way we update the sequence λ^k . Instead of using a classical projected gradient step with a scalar step size as in [10, 13, 17], we update λ^k using a projected weighted gradient step which allows us to obtain a distributed scheme. Further, we will analyze the convergence properties of Algorithm (**DFG**).

3.1 Sublinear convergence using an average primal sequence

As we have stated before, in this section we are interested in characterizing the dual suboptimality and also the primal suboptimality and feasibility violation for

Algorithm **(DFG)**. Using Lemma 1 instead of the classical descent lemma we can obtain from Theorem 2 in [17] the following inequality, which will help us to establish the convergence properties of Algorithm **(DFG)**:

$$\begin{aligned} & \frac{(k+1)(k+2)}{4}d(\hat{\lambda}^k) \\ & \geq \max_{\lambda \in \mathbb{D}} -\frac{1}{2}\|\lambda\|_W^2 + \sum_{s=0}^k \frac{s+1}{2} [d(\lambda^s) + \langle \nabla d(\lambda^s), \lambda - \lambda^s \rangle] \quad \forall \lambda \in \mathbb{D}. \end{aligned} \quad (22)$$

The following theorem provides an estimate on the dual suboptimality for Algorithm **(DFG)**:

Theorem 2 *Let Assumption 1 hold and the sequences $(z^k, \hat{\lambda}^k, \lambda^k)_{k \geq 0}$ be generated by algorithm **(DFG)**. Then, an estimate on dual suboptimality for (2) is given by:*

$$f^* - d(\hat{\lambda}^k) \leq \frac{2\mathcal{R}^2}{(k+1)^2}, \quad (23)$$

where $\mathcal{R} = \mathcal{R}(0) = \max_{\lambda^* \in \Lambda^*} \|\lambda^*\|_W$ according to (11).

Proof Using the concavity of d and $\lambda = \lambda^*$ in (22) we get:

$$\frac{(k+1)(k+2)}{4}d(\hat{\lambda}^k) \geq -\frac{1}{2}\|\lambda^*\|_W^2 + \sum_{s=0}^k \frac{s+1}{2}d(\lambda^s).$$

Dividing now both sides by $\frac{(k+1)(k+2)}{4}$, rearranging the terms and taking into account that $d(\lambda^*) = f^*$, $(k+1)^2 \leq (k+1)(k+2)$ and the definition of \mathcal{R} we obtain (23). \square

We define further the following average sequence for the primal variables:

$$\hat{z}^k = \sum_{s=0}^k \frac{2(s+1)}{(k+1)(k+2)} z^s. \quad (24)$$

The next result gives an estimate on primal feasibility violation.

Theorem 3 *Under the assumptions of Theorem 2 and \hat{z}^k generated by (24), an estimate on primal feasibility violation for original problem (1) is given by:*

$$\left\| \begin{bmatrix} A\hat{z}^k - b \\ [C\hat{z}^k - c]_{\mathbb{R}_+^q} \end{bmatrix} \right\|_{W^{-1}} \leq \frac{8\mathcal{R}}{(k+1)^2}. \quad (25)$$

Proof Using (22), the convexity of f and the definitions of d and ∇d , we can write for any $\lambda \in \mathbb{D}$:

$$\max_{\lambda \in \mathbb{D}} -\frac{2}{(k+1)^2}\|\lambda\|_W^2 + \langle \lambda, G\hat{z}^k - g \rangle \leq d(\hat{\lambda}^k) - f(\hat{z}^k). \quad (26)$$

For the second term of the right-hand side we have:

$$\begin{aligned} d(\hat{\lambda}^k) - f(\hat{z}^k) & \leq d(\lambda^*) - f(\hat{z}^k) = \min_{z \in \mathbb{R}^n} f(z) + \langle \lambda^*, Gz - g \rangle - f(\hat{z}^k) \\ & \leq f(\hat{z}^k) + \langle \lambda^*, G\hat{z}^k - g \rangle - f(\hat{z}^k) = \langle \lambda^*, G\hat{z}^k - g \rangle \leq \langle \lambda^*, [G\hat{z}^k - g]_{\mathbb{D}} \rangle, \end{aligned} \quad (27)$$

where in the last inequality we use that $\lambda^* \in \mathbb{D}$. By evaluating the maximum in the left-hand side term in (26) and taking into account that $\langle [G\hat{z}^k - g]_{\mathbb{D}}, G\hat{z}^k - g - [G\hat{z}^k - g]_{\mathbb{D}} \rangle = 0$ we obtain the following relation:

$$\max_{\lambda \in \mathbb{D}} -\frac{2}{(k+1)^2} \|\lambda\|_W^2 + \langle \lambda, G\hat{z}^k - g \rangle = \frac{(k+1)^2}{8} \|[G\hat{z}^k - g]_{\mathbb{D}}\|_{W^{-1}}^2. \quad (28)$$

Combining now (27) and (28) with (26), using the Cauchy-Schwartz inequality and introducing the notation $\alpha = \|[G\hat{z}^k - g]_{\mathbb{D}}\|_{W^{-1}}$, we obtain the following second order inequality in α :

$$\frac{(k+1)^2}{8} \alpha^2 - \|\lambda^*\|_W \alpha \leq 0,$$

from which, using the definitions of $G\hat{z}^k - g$, \mathbb{D} and \mathcal{R} we get (25). \square

Theorem 4 *Assume that the conditions in Theorem 3 are satisfied and let \hat{z}^k be given by (24). Then, the following estimate on primal suboptimality for problem (1) can be derived:*

$$-\frac{8\mathcal{R}^2}{(k+1)^2} \leq f(\hat{z}^k) - f^* \leq 0. \quad (29)$$

Moreover, the sequence \hat{z}^k converges to the unique optimal solution z^ of (1) with the the following rate:*

$$\|\hat{z}^k - z^*\| \leq \frac{4\mathcal{R}}{\sqrt{\sigma_f}(k+1)}. \quad (30)$$

Proof The right-hand side inequality in (29) follows from evaluating (26) in $\lambda = 0$ and taking into account that $d(\hat{\lambda}^k) \leq d(\lambda^*) = f^*$.

In order to prove the left-hand side inequality of (29) we can write:

$$\begin{aligned} f^* = d(\lambda^*) &= \min_{z \in \mathbb{R}^n} f(z) + \langle \lambda^*, Gz - g \rangle \\ &\leq f(\hat{z}^k) + \langle \lambda^*, G\hat{z}^k - g \rangle \\ &\leq f(\hat{z}^k) + \langle \lambda^*, [G\hat{z}^k - g]_{\mathbb{D}} \rangle \\ &\leq f(\hat{z}^k) + \|\lambda^*\|_W \|[G\hat{z}^k - g]_{\mathbb{D}}\|_{W^{-1}}, \end{aligned} \quad (31)$$

where the second inequality follows from the fact that $\lambda^* \in \mathbb{D}$ and the last one from Cauchy-Schwartz inequality. Using now (25) we obtain the result.

Further, since f is σ_f -strongly convex, we have also that $\mathcal{L}(z, \lambda)$ is also σ_f -strongly convex for all $\lambda \in \mathbb{D}$. Thus, taking $\lambda = \lambda^*$ and noting that $z^* = \arg \min_{z \in \mathbb{R}^n} \mathcal{L}(z, \lambda^*)$ we have:

$$\begin{aligned} \frac{\sigma_f}{2} \|\hat{z}^k - z^*\|^2 &\leq \mathcal{L}(\hat{z}^k, \lambda^*) - \mathcal{L}(z^*, \lambda^*) \\ &= f(\hat{z}^k) + \langle \lambda^*, G\hat{z}^k - g \rangle - f^* \stackrel{(29)}{\leq} \langle \lambda^*, G\hat{z}^k - g \rangle \\ &\leq \langle \lambda^*, [G\hat{z}^k - g]_{\mathbb{D}} \rangle \leq \|\lambda^*\|_W \|[G\hat{z}^k - g]_{\mathbb{D}}\|_{W^{-1}}, \end{aligned}$$

where the last two inequalities follows from the same arguments as in (31). Using now (25) and the definition of \mathcal{R} we obtain (30). \square

Remark 1 (i) If we use for the initialization of the algorithm any $\lambda^0 \in \mathbb{D}$ the order of the estimates on primal and dual suboptimality and primal feasibility violation derived above remain unchanged.

(ii) Note that according to Theorem 4 for $\lambda^0 = 0$ we are always below the optimal value f^* . In the case when we use an initialization $\lambda^0 \neq 0$ we cannot guarantee anymore this property.

(iii) From previous theorems we observe that for a given accuracy ϵ , we need to perform $\mathcal{O}\left(\frac{1}{\sqrt{\epsilon}}\right)$ iterations in order to obtain a primal suboptimal and near-feasible solution based on averaging the primal generated sequence. \square

4 Hybrid distributed dual fast gradient algorithm (H-DFG)

Note that for the Algorithm (DFG) the primal sequence $\{\hat{z}^k\}_{k \geq 0}$ for which we are able to recover primal suboptimality and primal feasibility violation is given by a weighted average of the iterates $\{z^k\}_{k \geq 0}$. However, in simulations we observe also a good behaviour of the last iterate z^k . In this section we propose a hybrid distributed dual fast gradient algorithm for which we can ensure estimates for both primal suboptimality and feasibility violation of the last iterate z^k , which supports our findings from simulations. The algorithm is characterized by two phases: in the first phase we perform k steps of Algorithm (DFG) while in the second phase another k steps of a dual weighted gradient algorithm are performed. A similar hybrid strategy was also discussed in [12, 18]. We present further the proposed scheme:

Algorithm (H-DFG)

Initialization: $\lambda^0 = 0$.

Phase 1: For $j = 0, \dots, k$ compute:

1. $z^j = \arg \min_{z \in \mathbb{R}^n} \mathcal{L}(z, \lambda^j)$
2. $\hat{\lambda}^j = [\lambda^j + W^{-1} \nabla d(\lambda^j)]_{\mathbb{D}}$
3. $\lambda^{j+1} = \frac{j+1}{j+3} \hat{\lambda}^j + \frac{2}{j+3} \left[W^{-1} \sum_{s=0}^j \frac{s+1}{2} \nabla d(\lambda^s) \right]_{\mathbb{D}}.$

Phase 2: Set $\lambda^k = \hat{\lambda}^k$. For $j = k, \dots, 2k$ compute:

1. $z^j = \arg \min_{z \in \mathbb{R}^n} \mathcal{L}(z, \lambda^j)$
2. $\lambda^{j+1} = [\lambda^j + W^{-1} \nabla d(\lambda^j)]_{\mathbb{D}}.$

The following lemma, which is a generalization of a standard result for gradient methods shows that Phase 2 of Algorithm (H-DFG) is an ascent method. For completeness we also give the proof.

Lemma 3 *Let the sequence $\{\lambda^j\}_j$ be generated by the Phase 2 of Algorithm (H-DFG). Then, the value of the dual function increases at each iteration according to the following relation:*

$$d(\lambda^{j+1}) \geq d(\lambda^j) + \frac{1}{2} \|\lambda^j - \lambda^{j+1}\|_W^2 \quad \forall j = k, \dots, 2k. \quad (32)$$

Proof Let us first notice that the update $\lambda^{j+1} = [\lambda^j + W^{-1}\nabla d(\lambda^j)]_{\mathbb{D}}$ can be posed as the minimization of the following quadratic approximation of d :

$$\lambda^{j+1} = \arg \max_{\lambda \in \mathbb{D}} d(\lambda^j) + \langle \nabla d(\lambda^j), \lambda - \lambda^j \rangle - \frac{1}{2} \|\lambda - \lambda^j\|_W^2. \quad (33)$$

From the optimality conditions of problem (33) we obtain:

$$\langle \nabla d(\lambda^j), \lambda^{j+1} - \lambda^j \rangle \geq \|\lambda^j - \lambda^{j+1}\|_W^2. \quad (34)$$

Using now this inequality in Lemma 1 with $\bar{\lambda} = \lambda^j$ and $\lambda = \lambda^{j+1}$ we obtain the result. \square

4.1 Sublinear convergence using the last primal iterate

We introduce further the following notation:

$$k^* = \arg \min_{j \in [k, 2k]} \|\lambda^j - \lambda^{j+1}\|_W^2. \quad (35)$$

Note that the quantity $\lambda^j - \lambda^{j+1}$ denotes the constrained gradient direction (see [16]), which represent an indicator for the suboptimality level of the estimate λ^j . We can also observe that λ^j is an optimal solution of (2) if and only if $\lambda^j - \lambda^{j+1} = 0$ and thus we want $\|\lambda^j - \lambda^{j+1}\|_W^2$ to be small. The following theorem gives an estimate on the dual suboptimality for the Algorithm (**H-DFG**):

Theorem 5 *Let Assumption 1 hold, the sequences $\{\lambda^j, \hat{\lambda}^j, z^j\}_{j \geq 0}$ be generated by the Algorithm (**H-DFG**) and k^* be given by (35). Then, an estimate for dual suboptimality for (2) is given by:*

$$f^* - d(\lambda^{k^*}) \leq \frac{2\mathcal{R}^2}{(k+1)^2}.$$

Proof From Theorem 2 and the initialization in Phase 2 of Algorithm (**H-DFG**) we have:

$$\frac{2\|\lambda^*\|_W^2}{(k+1)^2} \geq f^* - d(\hat{\lambda}^k) = f^* - d(\lambda^k).$$

Using now Lemma 3 we obtain the following inequalities:

$$d(\hat{\lambda}^k) = d(\lambda^k) \leq d(\lambda^{k+1}) \leq \dots \leq d(\lambda^{2k+1}), \quad (36)$$

from which, together with the previous inequality and the fact that $k^* \in [k, 2k]$ we obtain the result. \square

The following result characterizes the primal feasibility violation for Algorithm (**H-DFG**) in the last iterate z^{k^*} .

Theorem 6 *Under the assumptions of Theorem 5, an estimate on primal feasibility violation for original problem (1) is given by:*

$$\left\| \begin{bmatrix} Az^{k^*} - b \\ [Cz^{k^*} - c]_{\mathbb{R}_+^q} \end{bmatrix} \right\|_{W^{-1}} \leq \frac{2\mathcal{R}}{(k+1)\sqrt{(k+1)}}. \quad (37)$$

Proof Using Theorem 2 and Lemma 3 we can write:

$$\begin{aligned} \frac{2\|\lambda^*\|_W^2}{(k+1)^2} &\geq f^* - d(\hat{\lambda}^k) = f^* - d(\lambda^k) \\ &\geq f^* - d(\lambda^{k+1}) + \frac{1}{2}\|\lambda^k - \lambda^{k+1}\|_W^2 \geq \dots \geq f^* - d(\lambda^{2k+1}) + \frac{1}{2}\sum_{j=k}^{2k}\|\lambda^j - \lambda^{j+1}\|_W^2 \\ &\geq \frac{(k+1)}{2}\|\lambda^{k^*} - \lambda^{k^*+1}\|_W^2, \end{aligned}$$

where in the last inequality we used (35). Using the previous inequality we obtain:

$$\|\lambda^{k^*} - \lambda^{k^*+1}\|_W^2 \leq \frac{4\|\lambda^*\|_W^2}{(k+1)^3}. \quad (38)$$

Further, we will show that $\left\|\left[\nabla d(\lambda^{k^*})\right]_{\mathbb{D}}\right\|_{W^{-1}}^2 \leq \|\lambda^{k^*} - \lambda^{k^*+1}\|_W^2$. We will prove this inequality componentwise. First, we recall that $\mathbb{D} = \mathbb{R}^p \times \mathbb{R}_+^q$. Thus, for all $i = 1, \dots, p$ we have:

$$\begin{aligned} \left|\left[\nabla_i d(\lambda^{k^*})\right]_{\mathbb{R}_{W_{ii}^{-1}}}\right|^2 &= \left|\nabla_i d(\lambda^{k^*})\right|_{W_{ii}^{-1}}^2 = \left|\lambda_i^{k^*} - \lambda_i^{k^*+1} - W_{ii}^{-1}\nabla_i d(\lambda^{k^*})\right|_{W_{ii}}^2 \\ &= \left|\lambda_i^{k^*} - \lambda_i^{k^*+1}\right|_{W_{ii}}^2, \end{aligned} \quad (39)$$

where in the last inequality we used the definition of λ^{k^*+1} . We introduce now the following disjoint sets: $I_- = \{i \in [p+1, p+q] : \nabla_i d(\lambda^{k^*}) < 0\}$ and $I_+ = \{i \in [p+1, p+q] : \nabla_i d(\lambda^{k^*}) \geq 0\}$. Using these notations and the definition of \mathbb{D} , we can write for all $i \in I_-$:

$$\left|\left[\nabla_i d(\lambda^{k^*})\right]_{\mathbb{R}_+}\right|_{W_{ii}^{-1}}^2 = 0 \leq \left|\lambda_i^{k^*} - \lambda_i^{k^*+1}\right|_{W_{ii}}^2. \quad (40)$$

On the other hand, for all $i \in I_+$ we have:

$$\begin{aligned} \left|\left[\nabla_i d(\lambda^{k^*})\right]_{\mathbb{R}_+}\right|_{W_{ii}^{-1}}^2 &= \left|\nabla_i d(\lambda^{k^*})\right|_{W_{ii}^{-1}}^2 = \left|\left[W_{ii}^{-1}\nabla_i d(\lambda^{k^*})\right]_{\mathbb{R}_+}\right|_{W_{ii}}^2 \\ &= \left|\lambda_i^{k^*} - \lambda_i^{k^*+1}\right|_{W_{ii}}^2. \end{aligned} \quad (41)$$

Summing up the relations (39), (40) and (41) for all $i = 1, \dots, p+q$ and combine the result with (38) we obtain:

$$\left\|\left[\nabla d(\lambda^{k^*})\right]_{\mathbb{D}}\right\|_{W^{-1}}^2 \leq \|\lambda^{k^*} - \lambda^{k^*+1}\|_W^2 \leq \frac{4\|\lambda^*\|_W^2}{(k+1)^3}.$$

Taking now into account that $\left[\nabla d(\lambda^{k^*})\right]_{\mathbb{D}} = \begin{bmatrix} Az^{k^*} - b \\ [Cz^{k^*} - c]_{\mathbb{R}_+^q} \end{bmatrix}$ and using the

definition of \mathcal{R} we conclude the result. \square

We further characterize the primal suboptimality and also the distance from the last iterate z^{k^*} to the optimal solution z^* of the original optimization problem (1).

Theorem 7 *Let the conditions in Theorem 6 be satisfied and the function f be Lipschitz continuous with constant L_f , i.e. $|f(z) - f(y)| \leq L_f \|z - y\|$ for all $z, y \in \mathbb{R}^n$. Then, the following estimate on primal suboptimality for problem (1) can be derived:*

$$-\frac{2\mathcal{R}^2}{(k+1)\sqrt{(k+1)}} \leq f(z^{k*}) - f^* \leq \frac{2L_f\mathcal{R}}{\sqrt{\sigma_f}(k+1)}. \quad (42)$$

Moreover, the sequence z^{k*} converge to the unique optimal solution z^* of (1) with the the following rate:

$$\|z^{k*} - z^*\| \leq \frac{2\mathcal{R}}{\sqrt{\sigma_f}(k+1)}. \quad (43)$$

Proof The left-hand side inequality of (42) follows using a similar reasoning as in Theorem 4 and the result of Theorem 6. In order to prove the right hand-side inequality of (42) we first show (43). Using Lemma 2 with $\lambda = \lambda^{k*}$ we have:

$$\|z^{k*} - z^*\| \leq \sqrt{\frac{2}{\sigma_f}} \sqrt{f^* - d(\lambda^{k*})} \leq \frac{2\|\lambda^*\|_W}{\sqrt{\sigma_f}(k+1)},$$

with the last inequality resulting from Theorem 5. Using now the previous inequality and the Lipschitz property of f we obtain:

$$f(z^{k*}) - f^* \leq L_f \|z^{k*} - z^*\| \leq \frac{2L_f\|\lambda^*\|_W}{\sqrt{\sigma_f}(k+1)},$$

which together with the definition of \mathcal{R} conclude the statement. \square

Remark 2 (i) In a similar manner as in Section 3 using any $\lambda^0 \in \mathbb{D}$ for the initialization of the algorithm the order of estimates on primal and dual suboptimality and primal feasibility violation remain the same.

(ii) For a given accuracy ϵ , it follows from the results of this section that we need to perform $\mathcal{O}\left(\frac{1}{\sqrt[3]{\epsilon^2}}\right)$ iterations in order to be able to provide a primal suboptimal and near-feasible solutions based on the last primal iterate z^{k*} .

(iii) Even if the theoretical results show that the estimates on primal suboptimality and feasibility violation are worse for Algorithm **(H-DFG)** in comparison with the ones of Algorithm **(DFG)**, we have observed that in practice the last iterate behaves better. We will discuss this issue in more detail in Section 7.

Application of Algorithms **(DFG)** and **(H-DFG)** on practical engineering problems such as (DC-OPF) can be also found in [11].

5 Linear convergence for dual gradient method under an error bound property

In this section we show that under the additionally assumption that the gradients ∇f_i are Lipschitz continuous the dual problem (2) satisfies a certain error bound property which allows us to prove a global linear convergence for a distributed dual gradient method. From our best knowledge this is the first result showing global linear convergence of a dual gradient algorithm. All existing convergence results from the literature on dual gradient method either show sublinear convergence [1, 10, 13, 15] or at most *local* linear convergence [8].

5.1 Error bound property of the dual problem

In this section we assume that additionally we have Lipschitz continuity on the gradient of the primal objective function. Under strong convexity and this assumption we prove an error bound type property on the corresponding dual problem. Our approach for proving a certain error bound property is in a way similar to the one in [7, 26]. However, our results are more general in the sense that we allow the constraints set \mathbb{D} to be an unbounded polyhedron as opposed to the results in [26] where the authors show error bound property only for bounded polyhedra or the entire space. Also, our gradient mapping introduced below is more general than the one used in the standard analysis of the error bound property (see e.g. [7, 26]). Last but not least important is that our approach works also for dual problems. Thus, we make further the following assumption:

Assumption 8 *The convex functions f_i have Lipschitz continuous gradients w.r.t. Euclidean norm, with constants L_i .*

For the convex function f , we denote its conjugate by [22]:

$$\tilde{f}(y) = \sum_{i=1}^M \tilde{f}_i(y),$$

where $\tilde{f}_i(y) = \max_{x_i \in \mathbb{R}^{n_i}} \langle y, x_i \rangle - f_i(x_i)$. According to Proposition 12.60 in [22], under the Assumption 8 each function $\tilde{f}_i(y)$ is strongly convex w.r.t. Euclidean norm, with constant $\frac{1}{L_i}$, which implies that function \tilde{f} is strongly convex w.r.t. Euclidean norm, with constant:

$$\sigma_{\tilde{f}} = \min_{i \in \{1, \dots, M\}} \frac{1}{L_i}.$$

Note that in these settings our dual function can be written as:

$$d(\lambda) = -\tilde{f}(-G^T \lambda) - g^T \lambda. \quad (44)$$

The following lemma whose proof can be also found in [26, Lemma 4.2] or in [7, Lemma 3.1] will help us to prove the desired error bound property for our dual problem (2). For completeness we also give the proof.

Lemma 4 *Let Assumptions 1 and 8 hold. Then, there exists a unique $y^* \in \mathbb{R}^n$ such that:*

$$G^T \lambda^* = y^* \quad \forall \lambda^* \in \Lambda^*. \quad (45)$$

Moreover, $\nabla d(\lambda) = G \nabla \tilde{f}(-y^*) - g$ is constant for all $\lambda \in \Lambda$, where the set $\Lambda = \{\lambda \in \mathbb{D} : G^T \lambda = y^*\}$.

Proof Let $\lambda_1^*, \lambda_2^* \in \Lambda^*$. From concavity of d and the fact that the optimal value is the same for all $\lambda^* \in \Lambda^*$ we have:

$$d\left(\frac{\lambda_1^* + \lambda_2^*}{2}\right) = \frac{d(\lambda_1^*) + d(\lambda_2^*)}{2}.$$

Using now (44) we can write the following equality:

$$-f^*\left(-G^T \frac{\lambda_1^* + \lambda_2^*}{2}\right) - g^T \frac{\lambda_1^* + \lambda_2^*}{2} = -\frac{f^*(-G^T \lambda_1^*) + f^*(-G^T \lambda_2^*)}{2} - g^T \frac{\lambda_1^* + \lambda_2^*}{2}.$$

From the strong convexity property of \tilde{f} we have $G^T \lambda_1^* = G^T \lambda_2^*$, which implies that there exists a unique $y^* = G^T \lambda^*$ for all $\lambda^* \in \Lambda^*$. The second statement of the Lemma follows immediately from the definition of Λ and the fact that y^* is unique. \square

We introduce further the following notations:

$$r = [\lambda]_A^W, \bar{\lambda} = [\lambda]_{A^*}^W \text{ and } \bar{r} = [r]_{A^*}^W. \quad (46)$$

Remark 3 (i) Note that for any convex set D and any positive definite matrix W the projection mapping $[\cdot]_D^W$ is nonexpansive, i.e. $\|[\lambda]_D^W - [\omega]_D^W\|_W \leq \|\lambda - \omega\|_W$. In order to prove this nonexpansiveness property one can use a similar approach as for the Euclidean projection mapping $[\cdot]_D$.

(ii) In the case W is a positive definite diagonal matrix and the set D can be written as the Cartesian product of some sets in \mathbb{R} we have for any vector $\lambda \in D$ the following equivalence between projections $[\lambda]_D^W = [\lambda]_D$. \square

Using now the notations (46) we can write:

$$\|\lambda - \bar{\lambda}\|_W^2 \leq \|\lambda - \bar{r}\|_W^2 \leq (\|\lambda - r\|_W + \|r - \bar{r}\|_W)^2 \leq 2\|\lambda - r\|_W^2 + 2\|r - \bar{r}\|_W^2. \quad (47)$$

In what follows we will show how we can find upper bounds on $\|\lambda - r\|_W$ and $\|r - \bar{r}\|_W$ such that we will be able to establish an error bound property on the dual problem (2), i.e. there exists a positive constant κ , which depends on the original problem data L_i, σ_i, G and also on the norm $\|\lambda - \bar{\lambda}\|_W$ such that:

$$\|\lambda - \bar{\lambda}\|_W \leq \kappa(\|\lambda - \bar{\lambda}\|_W) \|\nabla^+ d(\lambda)\|_W \quad \forall \lambda \in \mathbb{D}, \quad (48)$$

where the mapping

$$\nabla^+ d(\lambda) = \left[\lambda + W^{-1} \nabla d(\lambda) \right]_{\mathbb{D}}^W - \lambda \quad (49)$$

denotes the gradient map. Further, we establish a result which will help us in proving the error bound (48):

Lemma 5 *Let Assumption 8 hold and $\nabla^+ d$ be given by (49). Then, the following inequality holds:*

$$\langle \nabla d(\omega) - \nabla d(\lambda), \lambda - \omega \rangle \leq 2\|\nabla^+ d(\lambda) - \nabla^+ d(\omega)\|_W \|\lambda - \omega\|_W \quad \forall \lambda, \omega \in \mathbb{D}.$$

Proof First, let us recall that $[\lambda + W^{-1} \nabla d(\lambda)]_{\mathbb{D}}^W$ is the unique solution of the optimization problem:

$$\min_{\xi \in \mathbb{D}} \|\xi - \lambda - W^{-1} \nabla d(\lambda)\|_W^2, \quad (50)$$

for which the optimality conditions reads:

$$\langle W \left([\lambda + W^{-1} \nabla d(\lambda)]_{\mathbb{D}}^W - (\lambda + W^{-1} \nabla d(\lambda)) \right), \xi - [\lambda + W^{-1} \nabla d(\lambda)]_{\mathbb{D}}^W \rangle \geq 0 \quad \forall \xi \in \mathbb{D}.$$

Taking now $\xi = [\omega + W^{-1} \nabla d(\omega)]_{\mathbb{D}}^W$ in the previous inequality, adding and subtracting λ and ω in the left term of the scalar product and using the definition of $\nabla^+ d$ we obtain:

$$\langle W \left(\nabla^+ d(\lambda) - W^{-1} \nabla d(\lambda) \right), \nabla^+ d(\lambda) + \lambda - \omega - \nabla^+ d(\omega) \rangle \leq 0,$$

which, together with the fact that W is symmetric implies:

$$\langle \nabla^+ d(\lambda) - W^{-1} \nabla d(\lambda), W(\lambda - \omega) + W(\nabla^+ d(\lambda) - \nabla^+ d(\omega)) \rangle \leq 0.$$

Rearranging now the terms in the previous inequality we have:

$$\begin{aligned} -\langle \nabla d(\lambda), \lambda - \omega \rangle &\leq -\langle \nabla^+ d(\lambda), W(\lambda - \omega) \rangle + \langle \nabla d(\lambda), \nabla^+ d(\lambda) - \nabla^+ d(\omega) \rangle \\ &\quad - \langle \nabla^+ d(\lambda), W(\nabla^+ d(\lambda) - \nabla^+ d(\omega)) \rangle. \end{aligned}$$

Writing now the previous inequality with λ and ξ interchanged and summing them up we can write:

$$\begin{aligned} \langle \nabla d(\xi) - \nabla d(\lambda), \lambda - \xi \rangle &\leq \langle \nabla^+ d(\xi) - \nabla^+ d(\lambda), W(\lambda - \xi) \rangle \\ &\quad + \langle \nabla d(\lambda) - \nabla d(\xi), \nabla^+ d(\lambda) - \nabla^+ d(\xi) \rangle - \|\nabla^+ d(\lambda) - \nabla^+ d(\xi)\|_W^2 \\ &\leq \langle \nabla^+ d(\xi) - \nabla^+ d(\lambda), W(\lambda - \xi) \rangle + \langle \nabla d(\lambda) - \nabla d(\xi), \nabla^+ d(\lambda) - \nabla^+ d(\xi) \rangle \\ &\leq \|\nabla^+ d(\lambda) - \nabla^+ d(\xi)\|_W (\|W(\lambda - \xi)\|_{W^{-1}} + \|\nabla d(\lambda) - \nabla d(\xi)\|_{W^{-1}}) \\ &\leq 2\|\nabla^+ d(\lambda) - \nabla^+ d(\xi)\|_W \|\lambda - \xi\|_W, \end{aligned}$$

which concludes the statement. \square

The next lemma gives an upper bound on $\|\lambda - r\|_W$:

Lemma 6 *Under the Assumptions 1 and 8 there exists a positive constant κ_1 such that the following inequality holds:*

$$\|\lambda - r\|_W^2 \leq \kappa_1 \|\nabla^+ d(\lambda)\|_W \|\lambda - \bar{\lambda}\|_W \quad \forall \lambda \in \mathbb{D}, \quad (51)$$

where $\kappa_1 = \frac{2}{\sigma_{\tilde{f}}^2} \theta_1^2$, with θ_1 being a finite constant depending on the matrix G .

Proof First, let us notice that we can write the set Λ as the following set characterized by linear equalities and inequalities:

$$\Lambda = \left\{ \omega \in \mathbb{R}^{p+q} : F\omega \leq 0, \quad G^T \omega = y^* \right\}, \quad (52)$$

where $F = [0_{q,p} - I_q]$. Since $\lambda \in \mathbb{D}$ this implies $F\lambda \leq 0$ and therefore, according to Theorem 2 in [21] we can write:

$$\|\lambda - r\|_W \leq \theta_1 \|G^T \lambda - y^*\|_\infty \leq \theta_1 \|G^T \lambda - y^*\|, \quad (53)$$

where θ_1 is finite and depends only on the matrix G and on the norms $\|\cdot\|_W$ and $\|\cdot\|_\infty$. From the strong convexity property of \tilde{f} combined with the fact that $G^T \bar{\lambda} = y^*$ we have:

$$\begin{aligned} \sigma_{\tilde{f}} \|G^T \lambda - y^*\|^2 &\leq \langle \nabla \tilde{f}(-G^T \lambda) - \nabla \tilde{f}(-G^T \bar{\lambda}), -G^T \lambda + G^T \bar{\lambda} \rangle \\ &= \langle -G \nabla \tilde{f}(-G^T \lambda) + g + G \nabla \tilde{f}(-G^T \bar{\lambda}) - g, \lambda - \bar{\lambda} \rangle \\ &= \langle \nabla d(\bar{\lambda}) - \nabla d(\lambda), \lambda - \bar{\lambda} \rangle \\ &\leq 2\|\nabla^+ d(\lambda) - \nabla^+ d(\bar{\lambda})\|_W \|\lambda - \bar{\lambda}\|_W = 2\|\nabla^+ d(\lambda)\|_W \|\lambda - \bar{\lambda}\|_W, \end{aligned} \quad (54)$$

where the last inequality follows from Lemma 5 with $\lambda = \lambda$ and $\xi = \bar{\lambda}$ and the last equality is deduced from the fact that since $\bar{\lambda} \in \Lambda^*$ this implies that $\nabla^+ d(\bar{\lambda}) = 0$. Combining now (53) with (54) we obtain the result. \square

The following result establishes an upper bound on $\|r - \bar{r}\|_W$:

Lemma 7 *Let Assumptions 1 and 8 hold. Then, the following inequality is valid:*

$$\|r - \bar{r}\|_W^2 \leq \kappa_2 \|\nabla^+ d(\lambda)\|_W \|\lambda - \bar{\lambda}\|_W \quad \forall \lambda \in \mathbb{D}, \quad (55)$$

where r, \bar{r} are given by (46) and

$$\kappa_2 = 6\theta_2^2 \left(2 \max_{\lambda^* \in \Lambda^*} \|\lambda - \lambda^*\|_W^2 + 2 \|\nabla d(\bar{\lambda})\|_{W^{-1}}^2 \right) \left(1 + 3\theta_1^2 \frac{2}{\sigma_{\bar{f}}} \right),$$

with θ_2 is a constant depending on C , $\nabla d(\bar{\lambda})$ and y^* .

Proof Since $\Lambda^* \subseteq \Lambda \subseteq \mathbb{D}$ and $G^T \xi = y^*$ for all $\xi \in \Lambda$, then the dual problem (2) has the same optimal solutions as the following linear problem:

$$\arg \max_{\lambda \in \mathbb{D}} d(\lambda) = \arg \max_{\xi \in \Lambda} d(\xi) = \arg \max_{\xi \in \Lambda} -\tilde{f}(-y^*) - \langle g, \xi \rangle = \arg \max_{\xi \in \Lambda} -\langle g, \xi \rangle. \quad (56)$$

Further, let us recall that $\nabla d(\bar{\lambda}) = G\tilde{f}(-y^*) - g$ for any $\lambda \in \mathbb{D}$ and thus we have that $\langle \nabla d(\bar{\lambda}), \xi \rangle = -\langle \nabla \tilde{f}(-y^*), y^* \rangle - \langle g, \xi \rangle$ for all $\xi \in \Lambda$. Therefore, we can write further:

$$\arg \max_{\xi \in \Lambda} \langle \nabla d(\bar{\lambda}), \xi \rangle = \arg \max_{\xi \in \Lambda} -\langle \nabla \tilde{f}(-y^*), y^* \rangle - \langle g, \xi \rangle = \arg \max_{\xi \in \Lambda} -\langle g, \xi \rangle. \quad (57)$$

Combining now (56) with (57) we can conclude that any solution $\bar{\xi} = [\xi]_{\Lambda^*}^W$ with $\xi \in \Lambda$ of the dual problem (2) is also a solution of problem (57). Since $\nabla_\nu d(\bar{\lambda}) = Az^* - b = 0$ and $\nabla_\mu d(\bar{\lambda}) = Cz^* - c \leq 0$, then we also have that the maximum in (57) is finite and thus problem (57) is solvable. Applying now Theorem 2 in [21] to the optimality conditions of problem (57) and its dual we obtain:

$$\|\xi - \bar{\xi}\|_W \leq \theta_2 |\langle \nabla d(\bar{\lambda}), \xi \rangle - \langle \nabla d(\bar{\lambda}), \bar{\xi} \rangle| \quad \forall \xi \in \Lambda, \quad (58)$$

where θ_2 is a constant depending only on the matrix C and on vectors $\nabla d(\bar{\lambda})$ and y^* (see eq. (6) in [21] for details). Using the previous relation we have:

$$\|\xi - \bar{\xi}\|_W \leq \theta_2 |\langle \nabla d(\bar{\lambda}), \xi \rangle - \langle \nabla d(\bar{\lambda}), \bar{\xi} \rangle| = \theta_2 \langle \nabla d(\bar{\lambda}), \bar{\xi} - \xi \rangle. \quad (59)$$

For any $\xi \in \Lambda$ the optimality conditions of the following projection problem $\min_{\omega \in \Lambda} \|\omega - \xi - W^{-1} \nabla d(\bar{\lambda})\|_W^2$ become:

$$\langle W \left(\left[\xi + W^{-1} \nabla d(\bar{\lambda}) \right]_A^W - \xi - W^{-1} \nabla d(\bar{\lambda}) \right), \left[\xi + W^{-1} \nabla d(\bar{\lambda}) \right]_A^W - \omega \rangle \leq 0$$

for all $\omega \in \Lambda$. Taking now $\omega = \bar{\xi} = [\xi]_{\Lambda^*}^W$ and since W is a symmetric matrix we obtain:

$$\begin{aligned} \langle \nabla d(\bar{\lambda}), \bar{\xi} - \xi \rangle &\leq \langle \left[\xi + W^{-1} \nabla d(\bar{\lambda}) \right]_A^W - \xi, W \left(\bar{\xi} - \left[\xi + W^{-1} \nabla d(\bar{\lambda}) \right]_A^W \right) + \nabla d(\bar{\lambda}) \rangle \\ &= \langle \left[\xi + W^{-1} \nabla d(\bar{\lambda}) \right]_A^W - \xi, W \left(\xi - \left[\xi + W^{-1} \nabla d(\bar{\lambda}) \right]_A^W \right) + W(\bar{\xi} - \xi) + \nabla d(\bar{\lambda}) \rangle \\ &\leq \langle \left[\xi + W^{-1} \nabla d(\bar{\lambda}) \right]_A^W - \xi, W(\bar{\xi} - \xi) + \nabla d(\bar{\lambda}) \rangle \\ &\leq \left\| \left[\xi + W^{-1} \nabla d(\bar{\lambda}) \right]_A^W - \xi \right\|_W (\|W(\bar{\xi} - \xi)\|_{W^{-1}} + \|\nabla d(\bar{\lambda})\|_{W^{-1}}) \\ &= \|\nabla^+ d(\xi)\|_W (\|\xi - \bar{\xi}\|_W + \|\nabla d(\bar{\lambda})\|_{W^{-1}}), \end{aligned}$$

where in the last equality we used the definition of $\nabla^+ d$ and the fact that $\nabla d(\bar{\lambda}) = \nabla d(\xi)$ for all $\xi \in \Lambda$ (see Lemma 4). Combining now the previous inequality with (59), taking $\xi = r \in \Lambda$ and squaring both sides we obtain:

$$\|r - \bar{r}\|_W^2 \leq \theta_2^2 (\|r - \bar{r}\|_W + \|\nabla d(\bar{\lambda})\|_{W^{-1}})^2 \|\nabla^+ d(r)\|_W^2. \quad (60)$$

Since $\bar{r} = [r]_{\Lambda^*}^W$ and $\Lambda^* \subseteq \Lambda$ we also have $\bar{r} = [\bar{r}]_{\Lambda}^W$. Thus, using the nonexpansive property of the projection we can write:

$$\|r - \bar{r}\|_W \leq \|\lambda - \bar{r}\|_W \leq \max_{\lambda^* \in \Lambda^*} \|\lambda - \lambda^*\|_W. \quad (61)$$

Further, our goal is to find an upper bound for $\|\nabla^+ d(r)\|_W$ in terms of $\|\lambda - \bar{\lambda}\|_W$ and $\|\nabla^+ d(\lambda)\|_W$. For this purpose let us first prove that $\nabla^+ d$ is Lipschitz continuous with constant 3 w.r.t to the norm $\|\cdot\|_W$. For any $\lambda, \tilde{\lambda} \in \mathbb{D}$ we can write:

$$\begin{aligned} \|\nabla^+ d(\lambda) - \nabla^+ d(\tilde{\lambda})\|_W &\leq \|\lambda - \tilde{\lambda}\|_W + \left\| \left[\lambda + W^{-1} \nabla d(\lambda) \right]_{\mathbb{D}}^W - \left[\tilde{\lambda} + W^{-1} \nabla d(\tilde{\lambda}) \right]_{\mathbb{D}}^W \right\| \\ &\leq \|\lambda - \tilde{\lambda}\|_W + \|\lambda + W^{-1} \nabla d(\lambda) - \tilde{\lambda} - W^{-1} \nabla d(\tilde{\lambda})\|_W \\ &\leq 2\|\lambda - \tilde{\lambda}\|_W + \|\nabla d(\lambda) - \nabla d(\tilde{\lambda})\|_{W^{-1}} \leq 3\|\lambda - \tilde{\lambda}\|_W. \end{aligned} \quad (62)$$

Using now (62) with $\tilde{\lambda} = \bar{\lambda}$ and taking into account that $\nabla^+ d(\bar{\lambda}) = 0$, we have:

$$\|\nabla^+ d(\lambda)\|_W = \|\nabla^+ d(\lambda) - \nabla^+ d(\bar{\lambda})\|_W \leq 3\|\lambda - \bar{\lambda}\|_W. \quad (63)$$

Using now again (62) and the Lipschitz continuity property of ∇d we can write:

$$\begin{aligned} \|\nabla^+ d(r)\|_W^2 &\leq \left(\|\nabla^+ d(\lambda)\|_W + \|\nabla^+ d(r) - \nabla^+ d(\lambda)\|_W \right)^2 \\ &\leq 2\|\nabla^+ d(\lambda)\|_W^2 + 2\|\nabla^+ d(r) - \nabla^+ d(\lambda)\|_W^2 \\ &\leq 6\|\nabla^+ d(\lambda)\|_W \|\lambda - \bar{\lambda}\|_W + 18\|\lambda - r\|_W^2 \\ &\leq 6 \left[1 + 3\theta_1^2 \frac{2}{\sigma_{\tilde{f}}} \right] \|\nabla^+ d(\lambda)\|_W \|\lambda - \bar{\lambda}\|_W, \end{aligned} \quad (64)$$

where in the last inequality we used (51). Introducing now (61) and (64) in (60) and using the inequality $(\alpha + \beta)^2 \leq 2\alpha^2 + 2\beta^2$ we obtain the result. \square

Note that for any finite $\lambda \in \mathbb{D}$, since Λ^* is bounded we have that $\max_{\lambda^* \in \Lambda^*} \|\lambda - \lambda^*\|_W$ is also finite. The following theorem establishes an error bound type property for the dual problem (2):

Theorem 9 *Let Assumptions 1 and 8 hold. Then, there exists κ , depending on the data of the original problem and $\max_{\lambda^* \in \Lambda^*} \|\lambda - \lambda^*\|_W$, such that the following error bound property can be established for the dual problem (2):*

$$\|\lambda - \bar{\lambda}\|_W \leq \kappa (\|\lambda - \lambda^*\|_W) \|\nabla^+ d(\lambda)\|_W \quad \forall \lambda \in \mathbb{D}, \quad (65)$$

with:

$$\kappa (\|\lambda - \lambda^*\|_W) = \kappa_1 + \kappa_2 = \theta_1^2 \frac{4}{\sigma_{\tilde{f}}} + 12\theta_2^2 \left(2 \max_{\lambda^* \in \Lambda^*} \|\lambda - \lambda^*\|_W^2 + 2\|\nabla d(\bar{\lambda})\|_W^2 \right) \left(1 + 3\theta_1^2 \frac{2}{\sigma_{\tilde{f}}} \right).$$

Proof The result follows immediately by using (51) from Lemma 6 and (55) from Lemma 7 in (47) and dividing both sides by $\|\lambda - \bar{\lambda}\|_W$. \square

5.2 Convergence analysis using the last iterate

Under the error bound property for the dual problem defined in Theorem 9 we will show in this section linear convergence for a distributed dual gradient method. From our knowledge this is the first result showing *global* linear convergence rate on primal suboptimality and infeasibility for the last primal iterate of a dual gradient algorithm, as opposed to the results in [8] where only *local* linear convergence was derived for such an algorithm. Thus, we now introduce the following distributed dual gradient method:

Algorithm (DG)

Initialization: $\lambda^0 = 0$. For $k \geq 0$ compute:

1. $z^k = \arg \min_{z \in \mathbb{R}^n} \mathcal{L}(z, \lambda^k)$.
2. $\lambda^{k+1} = [\lambda^k + W^{-1} \nabla d(\lambda^k)]_{\mathbb{D}}$.

The next lemma, which is a generalization of a known result for the gradient method (see e.g. [10, 16]) will help us to analyze the convergence of the Algorithm (DG):

Lemma 8 *Let Assumption 1 hold and the sequence $\{\lambda^k\}_{k \geq 0}$ be generated by Algorithm (DG). Then, the following inequalities hold:*

$$\|\lambda^k - \lambda^*\|_W \leq \dots \leq \|\lambda^0 - \lambda^*\|_W \quad \forall \lambda^* \in \Lambda^*, k \geq 0. \quad (66)$$

Proof Taking $\lambda = \lambda^*$ in the optimality condition of (33), we obtain the following inequality:

$$\langle \nabla d(\lambda^k) - W(\lambda^{k+1} - \lambda^k), \lambda^* - \lambda^{k+1} \rangle \leq 0. \quad (67)$$

Further, we can write:

$$\begin{aligned} \|\lambda^{k+1} - \lambda^*\|_W^2 &= \|\lambda^{k+1} - \lambda^k + \lambda^k - \lambda^*\|_W^2 \\ &= \|\lambda^k - \lambda^*\|_W^2 + 2\langle W(\lambda^{k+1} - \lambda^k), \lambda^k - \lambda^{k+1} + \lambda^{k+1} - \lambda^* \rangle + \|\lambda^{k+1} - \lambda^k\|_W^2 \\ &= \|\lambda^k - \lambda^*\|_W^2 + 2\langle W(\lambda^{k+1} - \lambda^k), \lambda^{k+1} - \lambda^* \rangle - \|\lambda^{k+1} - \lambda^k\|_W^2 \\ &\leq \|\lambda^k - \lambda^*\|_W^2 - 2\langle \nabla d(\lambda^k), \lambda^* - \lambda^k \rangle \\ &\quad + 2\left(\langle \nabla d(\lambda^k), \lambda^{k+1} - \lambda^k \rangle - \frac{1}{2}\|\lambda^{k+1} - \lambda^k\|_W^2\right) \\ &\leq \|\lambda^k - \lambda^*\|_W^2 + 2\left(d(\lambda^k) - d(\lambda^*)\right) + 2\left(d(\lambda^{k+1}) - d(\lambda^k)\right) \\ &= \|\lambda^k - \lambda^*\|_W^2 + 2\left(d(\lambda^{k+1}) - d(\lambda^*)\right) \leq \|\lambda^k - \lambda^*\|_W^2, \end{aligned} \quad (68)$$

where the first inequality follows from (67) and the second one is derived from the concavity of the function d and Lemma 1. Applying now recursively the previous inequality we obtain (66). \square

Using now (66) and the definition of \mathcal{R} we can write for all $k \geq 0$:

$$\max_{\lambda^* \in \Lambda^*} \|\lambda^k - \lambda^*\|_W \leq \max_{\lambda^* \in \Lambda^*} \|\lambda^0 - \lambda^*\|_W = \mathcal{R}, \quad (70)$$

where the last equality follows from the definition of \mathcal{R} and the fact that $\lambda^0 = 0$. Introducing now this inequality in (65) we obtain:

$$\|\lambda^k - \bar{\lambda}^k\|_W \leq \bar{\kappa} \|\nabla^+ d(\lambda^k)\|_W \quad \forall k \geq 0, \quad (71)$$

where:

$$\bar{\kappa} = \theta_1^2 \frac{4}{\sigma_{\bar{\Gamma}}} + 12\theta_2^2 \left(2\mathcal{R}^2 + 2\|\nabla d(\bar{\lambda}^k)\|_W^2 \right) \left(1 + 3\theta_1^2 \frac{2}{\sigma_{\bar{\Gamma}}} \right).$$

Using now Remark 3 (ii) and the definition of $\nabla^+ d$ we have:

$$\|\nabla^+ d(\lambda^k)\|_W = \|\lambda^{k+1} - \lambda^k\|_W. \quad (72)$$

Further, combining (71) with (72) we can write:

$$\|\lambda^k - \bar{\lambda}^k\|_W \leq \bar{\kappa} \|\nabla^+ d(\lambda^k)\|_W = \bar{\kappa} \|\lambda^{k+1} - \lambda^k\|_W. \quad (73)$$

The following theorem provides an estimate on the dual suboptimality for Algorithm (DG):

Theorem 10 *Let Assumptions 1 and 8 hold and the sequences $(z^k, \lambda^k)_{k \geq 0}$ be generated by algorithm (DG). Then, an estimate on dual suboptimality for (2) is given by:*

$$f^* - d(\lambda^{k+1}) \leq \left(\frac{4(1 + \bar{\kappa})}{1 + 4(1 + \bar{\kappa})} \right)^k (f^* - d(\lambda^0)). \quad (74)$$

Proof First, let us notice that for any $k \geq 0$, λ^{k+1} can be computed as the unique optimal solution of problem (33). Thus, from the optimality condition of problem (33) we have:

$$\langle \nabla d(\lambda^k), \bar{\lambda}^k - \lambda^{k+1} \rangle \leq \langle W(\lambda^{k+1} - \lambda^k), \bar{\lambda}^k - \lambda^{k+1} \rangle \leq 0, \quad (75)$$

where we recall that $\bar{\lambda}^k = [\lambda^k]_{A^*}^W$. Further, since the optimal value of the dual function is unique we can write:

$$\begin{aligned} f^* - d(\lambda^{k+1}) &= d(\bar{\lambda}^k) - d(\lambda^{k+1}) \leq \langle \nabla d(\lambda^{k+1}), \bar{\lambda}^k - \lambda^{k+1} \rangle \\ &= \langle \nabla d(\lambda^{k+1}) - \nabla d(\lambda^k), \bar{\lambda}^k - \lambda^{k+1} \rangle + \langle \nabla d(\lambda^k), \bar{\lambda}^k - \lambda^{k+1} \rangle \\ &\leq \|\nabla d(\lambda^{k+1}) - \nabla d(\lambda^k)\|_{W^{-1}} \|\bar{\lambda}^k - \lambda^{k+1}\|_W \\ &\quad + \langle W(\lambda^{k+1} - \lambda^k), \bar{\lambda}^k - \lambda^{k+1} \rangle \\ &\leq \|\lambda^{k+1} - \lambda^k\|_W \|\bar{\lambda}^k - \lambda^{k+1}\|_W + \|\lambda^{k+1} - \lambda^k\|_W \|\bar{\lambda}^k - \lambda^{k+1}\|_W \\ &= 2\|\lambda^{k+1} - \lambda^k\|_W \|\bar{\lambda}^k - \lambda^{k+1}\|_W, \end{aligned} \quad (76)$$

where the second inequality follows from (75). Using now relation (73) we can write:

$$\|\bar{\lambda}^k - \lambda^{k+1}\|_W \leq \|\bar{\lambda}^k - \lambda^k\|_W + \|\lambda^k - \lambda^{k+1}\|_W \leq (1 + \bar{\kappa}) \|\lambda^k - \lambda^{k+1}\|_W.$$

Introducing now the previous inequality in (76) and using Lemma 3 we have:

$$f^* - d(\lambda^{k+1}) \leq 2(1 + \bar{\kappa}) \|\lambda^k - \lambda^{k+1}\|_W^2 \leq 4(1 + \bar{\kappa}) (d(\lambda^{k+1}) - d(\lambda^k)).$$

Rearranging the terms in the previous inequality we obtain:

$$f^* - d(\lambda^{k+1}) \leq \frac{4(1+\bar{\kappa})}{1+4(1+\bar{\kappa})} (f^* - d(\lambda^k)). \quad (77)$$

Applying now (77) recursively we obtain (74). \square

In order to characterize the dual suboptimality we extend the proof for the centralized gradient algorithm [8, 26] to the case of distributed dual gradient Algorithm (**DG**). The following theorem gives an estimate on the primal feasibility violation for Algorithm (**DG**):

Theorem 11 *Under the assumptions of Theorem 10, the following estimate holds for the primal feasibility violation:*

$$\left\| \begin{bmatrix} Az^k - b \\ [Cz^k - c]_{\mathbb{R}_+^q} \end{bmatrix} \right\|_{W^{-1}} \leq \left(\frac{4(1+\bar{\kappa})}{1+4(1+\bar{\kappa})} \right)^{\frac{k-1}{2}} \sqrt{2(f^* - d(\lambda^0))}. \quad (78)$$

Proof Using the descent property of dual gradient method (32) we have:

$$\begin{aligned} \|\lambda^k - \lambda^{k+1}\|_W^2 &\leq 2(d(\lambda^{k+1}) - d(\lambda^k)) \leq 2(f^* - d(\lambda^k)) \\ &\leq 2 \left(\frac{4(1+\bar{\kappa})}{1+4(1+\bar{\kappa})} \right)^{k-1} (f^* - d(\lambda^0)), \end{aligned} \quad (79)$$

where in the last inequality we used Theorem 10. Using now a similar reasoning as in Theorem 6, we obtain:

$$\left\| [\nabla d(\lambda^k)]_{\mathbb{D}} \right\|_{W^{-1}}^2 \leq \|\lambda^k - \lambda^{k+1}\|_W^2 \leq 2 \left(\frac{4(1+\bar{\kappa})}{1+4(1+\bar{\kappa})} \right)^{k-1} (f^* - d(\lambda^0)),$$

where in second inequality we used (79). Squaring now both sides of previous inequality and taking into account the definitions of ∇d and \mathbb{D} we obtain (78). \square

We now characterize the primal suboptimality and the distance from the last iterate z^k generated by Algorithm (**DG**) to the optimal solution z^* of our original optimization problem (1).

Theorem 12 *Let the conditions in Theorem 11 be satisfied. Then, the following estimate on primal suboptimality for problem (1) can be derived:*

$$- \left(\frac{4(1+\bar{\kappa})}{1+4(1+\bar{\kappa})} \right)^{\frac{k-1}{2}} \mathcal{R} \sqrt{2(f^* - d(\lambda^0))} \leq f(z^k) - f^* \leq v(k), \quad (80)$$

where

$$\begin{aligned} v(k) &= \frac{\mathcal{R}}{\underline{w}} \|G\| \left(\frac{4(1+\bar{\kappa})}{1+4(1+\bar{\kappa})} \right)^{\frac{k}{2}} \sqrt{\frac{2}{\sigma_f} (f^* - d(\lambda^0))} \\ &\quad + \frac{\max_{i=1,\dots,M} L_i}{2} \left(\frac{4(1+\bar{\kappa})}{1+4(1+\bar{\kappa})} \right)^k \frac{2}{\sigma_f} (f^* - d(\lambda^0)). \end{aligned}$$

Moreover, the sequence z^k converge to the unique optimal solution z^* of (1) with the the following rate:

$$\|z^k - z^*\| \leq \left(\frac{4(1+\bar{\kappa})}{1+4(1+\bar{\kappa})} \right)^{\frac{k}{2}} \sqrt{\frac{2}{\sigma_f} (f^* - d(\lambda^0))}. \quad (81)$$

Proof The left-hand side inequality of (80) follows using a similar reasoning as in Theorem 4 and the result of Theorem 11. In order to prove the right hand-side inequality of (80) we first show (81). Using Lemma 2 with $\lambda = \lambda^k$ we have:

$$\|z^k - z^*\| \leq \sqrt{\frac{2}{\sigma_f}} \sqrt{f^* - d(\lambda^{k*})} \leq \left(\frac{4(1 + \bar{\kappa})}{1 + 4(1 + \bar{\kappa})} \right)^{\frac{k}{2}} \sqrt{\frac{2}{\sigma_f}} (f^* - d(\lambda^0)),$$

with the last inequality resulting from Theorem 10. Let us introduce further the notation by $\underline{w} = \lambda_{\min}(W)$. Using now the continuous Lipschitz property of ∇f we obtain:

$$\begin{aligned} f(z^k) - f^* &\leq \langle \nabla f(z^*), z^k - z^* \rangle + \frac{\max_i L_i}{2} \|z^k - z^*\|^2 \\ &= \langle -G^T \lambda^*, z^k - z^* \rangle + \frac{\max_i L_i}{2} \|z^k - z^*\|^2 \\ &\leq \frac{\mathcal{R} \|G\|}{\underline{w}} \|z^k - z^*\| + \frac{\max_i L_i}{2} \|z^k - z^*\|^2, \end{aligned}$$

where the first equality is deduced from the optimality conditions of problem $z^* = \arg \min f(z) + \langle \lambda^*, Gz - g \rangle$ and in the last inequality we used Cauchy-Schwartz inequality, the fact that $\|\cdot\| \leq \frac{1}{\underline{w}} \|\cdot\|_W$ and the definition of \mathcal{R} . Using now (81) in the previous inequality we obtain the result. \square

6 Distributed implementation

In this section we analyze the distributed implementation of Algorithms **(DFG)**, **(H-DFG)** and **(DG)**. We look first at step 1 of the Algorithm **(DFG)**. Note that this step is similar with the steps 1 of phases 1 and 2 of Algorithm **(H-DFG)** and the step 1 of Algorithm **(DG)** and therefore their analysis follows in a similar way. According to (6), for all $i \in V_1$ we have:

$$\begin{aligned} z_i^k &= \arg \min_{z_i \in \mathbb{R}^{n_i}} f_i(z_i) + \left\langle \lambda^k, \left[A_i^T C_i^T \right]^T z_i \right\rangle \\ &= \arg \min_{z_i \in \mathbb{R}^{n_i}} f_i(z_i) + \sum_{j \in \mathcal{N}_i} \left(\left[A_{ji}^T C_{ji}^T \right] \lambda_j^k \right)^T z_i. \end{aligned} \quad (82)$$

Thus, in order to compute z_i^k the algorithm requires only local information, namely $\{A_{ji}, C_{ji}, \lambda_j^k\}_{j \in \mathcal{N}_i}$. For example, in the case of (NUM) problem, the update of source rate z_i^k requires only the link prices λ_j^k which are utilized by source i . Using now the definitions of W and ∇d , step 2 in Algorithm **(DFG)** can be written in the following form:

$$\hat{\lambda}_j^k = \left[\lambda_j^k + \left[\begin{array}{c} W_{\nu jj}^{-1} \sum_{i \in \mathcal{N}_j} A_{ji} z_i^k \\ W_{\mu jj}^{-1} \sum_{i \in \mathcal{N}_j} C_{ji} z_i^k \end{array} \right] \right]_{\mathbb{R}^{p_j} \times \mathbb{R}_+^{q_j}}, \quad \forall j \in V_2, \quad (83)$$

where $W_{\nu jj}$ and $W_{\mu jj}$ denote the j th block-diagonal element of matrix W_ν and W_μ , respectively. Taking into account the definitions of $W_{\nu jj}$ and $W_{\mu jj}$ we can conclude that in order to update the dual variable $\hat{\lambda}_j^k$ in step 2 of Algorithm

(**DFG**) we require only local information $\{L_{d_i}, A_{ji}, C_{ji}, z_i^k\}_{i \in \bar{\mathcal{N}}_j}$. Thus, in the case of (NUM) problem, the update of the link price $\hat{\lambda}_j^k$ requires only the source rates z_i^k which use link j . Note that analysis of step 3 in the Algorithm (**DFG**) can be derived in a similar way as for step 2. Also, step 2 in phases 1 and 2 and step 3 in phase 1 of the Algorithm (**H-DFG**) follows similarly. Note also that Algorithm (**DG**) has the same iterations as phase 2 of Algorithm (**H-DFG**).

Further, we note that all the estimates for the convergence rate for primal and dual suboptimality and primal feasibility violation derived in Sections 3.1, 4.1 and 5.2 depends on the upper bound on the norm of the optimal Lagrange multipliers \mathcal{R} , which at its turn depends on the degree of separability of problem (1), characterized by the sets \mathcal{N}_i and $\bar{\mathcal{N}}_j$. In order to see this dependence we can write further:

$$\mathcal{R}^2 = \max_{\lambda^* \in \Lambda^*} \|\lambda^*\|_W^2 = \max_{\lambda^* \in \Lambda^*} \sum_{j=1}^{\bar{M}} \sum_{i \in \bar{\mathcal{N}}_j} L_{d_i} \|\lambda_j^*\|^2, \quad (84)$$

from which it is straightforward to notice that \mathcal{R} depends on the cardinality of each $\bar{\mathcal{N}}_j$. On the other hand, for each i we recall that:

$$L_{d_i} = \frac{\left\| \begin{bmatrix} [A_{ji}]_{j \in \mathcal{N}_i} \\ [C_{ji}]_{j \in \mathcal{N}_i} \end{bmatrix} \right\|^2}{\sigma_i},$$

which depends on the cardinality of the set \mathcal{N}_i . Thus, we can conclude that \mathcal{R} depends on the cardinality of \mathcal{N}_i and $\bar{\mathcal{N}}_j$ which represent a natural measure for the degree of separability of our original problem (1).

7 Numerical simulations

In order to certify the theoretical results previously presented, in this section we test the performances of Algorithms (**DFG**), (**H-DFG**) and distributed dual gradient Algorithm (**DG**) for solving the (DC-OPF) problem in form (19) for different IEEE bus test cases. We recall that in the Algorithm (**DG**), at each iteration k the dual variable is updated as follows:

$$\lambda^{k+1} = \left[\lambda^k + W^{-1} \nabla d(\lambda^k) \right]_{\mathbb{D}}.$$

The numerical simulation are performed on different power systems, representing classical test cases from the literature [28], with the number of buses M ranging from 9 to 300, the number of generators from 3 to 69 and the number of interconnecting lines from 18 to 411. The descriptions of the power systems are listed in the table below:

For each power system considered for simulation we generate the local constraints sets imposed on the phase angle and on the generated power of each bus i , Θ_i and \mathcal{P}_i , respectively, the local loads P_i^d and the matrices E , R and A^g using the data extract from the MATPOWER toolbox [28]. Also, for each test case we take the parameters of the local cost functions as follows: $q_i = 2$, $p_i = 10$, $\gamma_i = 2$ and $\beta_i = 0.1$.

Test case	M	M_g	M	n	p	q	Details
pws ₁	9	3	9	12	9	18	Example taken from [28]
pws ₂	14	5	20	19	14	40	IEEE 14 bus test case
pws ₃	30	6	41	36	30	82	IEEE 30 bus test case
pws ₄	39	10	46	49	39	92	39 bus New England system
pws ₅	57	7	80	64	57	160	IEEE 57 bus test case
pws ₆	118	54	186	172	118	372	IEEE 118 bus test case
pws ₇	300	69	411	369	300	822	IEEE 300 bus test case

Table 1 Description of the test cases.

In the case of (DC-OPF) problem the Lagrangian function takes the following form:

$$\mathcal{L}(\theta, P^g, \lambda) = \sum_{i=1}^M f_i(\theta_i, P_i^g) + \langle \nu, E^T RE\theta - A^g P^g + P^d \rangle + \langle \mu, \begin{bmatrix} RE \\ -RE \end{bmatrix} \theta - \begin{bmatrix} -\bar{F} \\ \underline{F} \end{bmatrix} \rangle$$

where we recall that $\theta = [\theta_1^T \dots \theta_M^T]^T$, $P^g = [(P_1^g)^T \dots (P_M^g)^T]^T$, the functions f_i are given by (17) if the bus i is directly coupled to a generator unit or by (18) otherwise and $\lambda = [\nu^T \mu^T]^T$. For all algorithms, for each Lagrange multiplier λ we have to compute the optimal solution of the inner problem, i.e. the minimization of the Lagrange function subject to the local constraints $\theta_i \in \Theta_i$ and $P_i^g \in \mathcal{P}_i$. As we have shown in Section 6, due to the separability of Lagrangian \mathcal{L} , this can be done distributively, i.e. computing the phase angle $\theta_i(\lambda)$ and the generated power $P_i^g(\lambda)$, for a given trading price λ , require only local information. Moreover, in the case of (DC-OPF) problem (19), $\theta_i(\lambda)$ and $P_i^g(\lambda)$ can be computed in closed form by solving the following scalar equations derived from the optimality conditions of the inner problems:

$$\begin{cases} q_i (\theta_i - \theta_i^{\text{ref}}) + \sum_{j \in \mathcal{S}_i} \nu_j [E^T RE]_{ji} + \sum_{l \in \mathcal{N}_i} \mu_l^T \begin{bmatrix} [RE]_{li} \\ -[RE]_{li} \end{bmatrix} = 0 \\ p_i (P_i^g - P_i^{g, \text{ref}}) + \nu_i A_{ij_i}^g - \frac{\gamma_i}{\beta_i + P_i^g} = 0, \end{cases}$$

for all buses $i \in V_1$, where j_i denotes the position of the generator unit in P^g directly coupled to bus i . In order to compute $\theta_i^k = \theta_i(\lambda^k)$ and $P_i^{g^k} = P_i^g(\lambda^k)$ for an iteration k , after solving the previous equations we have to project their solutions onto the local box constraints sets Θ_i and \mathcal{P}_i .

The reader should note that in the context of (DC-OPF) problem, ν multipliers associated to the power balance equation have the economic interpretation as the optimal energy trading prices at the buses of the network. Therefore, our algorithms are able to identify also the optimal energy pricing rates for the energy traded through the interconnections in a distributed fashion. Thus, it is not necessary to set up a common control center, but it is sufficient to interchange a small amount of information among the involved buses. Moreover, the update of the trading prices (dual variables) can be also done in a distributed fashion as follows:

$$\begin{cases} \hat{\mu}_l^{k+1} = \mu_l^k + W_{\mu_{ll}}^{-1} \sum_{i \in \mathcal{N}_l} \left(\begin{bmatrix} [RE]_{li} \\ -[RE]_{li} \end{bmatrix} \theta_i^k - \begin{bmatrix} \bar{F}_l \\ -\underline{F}_l \end{bmatrix} \right) \\ \hat{\nu}_j^{k+1} = \nu_j^k + W_{\nu_{jj}}^{-1} \sum_{i \in \mathcal{S}_j} \left([E^T RE]_{ji} \theta_i^k - A_{ij_i}^g P_i^{g^k} + P_j^d \right), \end{cases}$$

for all lines $l \in V_2$ and buses $j \in V_1$. We solve the (DC-OPF) problem (19) using Algorithms **(DFG)**, **(H-DFG)** and **(DG)** and we compare their performances in terms of the number of iterations. We also consider the centralized versions of these algorithms, namely: **(CFG)**, **(H-CFG)** and **(CG)**, where by centralized version we understand the version of the algorithm where instead of the step size given by matrix W we use $L_d I_{p+q}$ with L_d denoting the Lipschitz constant of the gradient ∇d of the dual function. We recall that the optimization variable $z = [z_1^T \cdots z_M^T]^T$, where $z_i = \theta_i$ for the buses which do not have a generator unit and $z_i = [\theta_i^T \ P_i^g]^T$ for the ones directly coupled to a generator. In order to construct the matrix A we interpolate the columns of $E^T R E$ and A^g on the corresponding positions, while C is formed by intercalating in the matrix $[(RE)^T - (RE)^T]^T$ columns with elements equal to zero on the positions corresponding to the position of P_i^g in the vector x .

In Table 2 we show, for each test case, the number of iterations performed by the algorithms in order to find a suboptimal primal solution \hat{z}^k which satisfy the following stopping criteria for primal suboptimality and feasibility violation:

$$\frac{|f(\hat{z}^k) - f^*|}{f^*} \leq \epsilon \text{ and } \left\| \begin{bmatrix} G\hat{z}^k - g \end{bmatrix}_{\mathbb{D}} \right\|_{W^{-1}} \leq \epsilon, \quad (85)$$

where we recall that $G = [A^T \ C^T]^T$ and $g = [b^T \ c^T]^T$. Note that in the case of Algorithm **(DFG)** \hat{z}^k is given by (24), while for Algorithms **(H-DFG)** and **(DG)** $\hat{z}^k = z^{k*}$ and $\hat{z}^k = z(\lambda^k)$, respectively. We also consider the same estimates for the centralized version of the algorithms. In our simulation we consider an accuracy $\epsilon = 0.01$. It is straightforward to notice that the suboptimality criterion satisfied with this accuracy implies the fact that the difference between the value of the cost function $f(\hat{z}^k)$ and the optimal value f^* is less than 1%. For each test case, we use CVX in order to compute the optimal value f^* . Also, in the case when the imposed accuracy has not been attained after $3 \cdot 10^5$ iterations, we stoped the algorithm and reported *.

Test case \ Algorithm	DFG	CFG	H-DFG	H-CFG	DG	CG
pws ₁	4486	4134	700	646	168619	143283
pws ₂	1991	1920	944	1066	203210	214746
pws ₃	1368	2013	503	1356	27026	52893
pws ₄	1756	6343	1316	4835	69961	275343
pws ₅	4876	21123	2003	15507	*	*
pws ₆	8117	45787	5787	35624	*	*
pws ₇	19432	63456	9978	67843	*	*

Table 2 Number of iterations performed for finding an ϵ -suboptimal solution of the (DC-OPF) problem for each test case.

Some remarks are worth to be mentioned. First, we can observe from Table 2 that both the proposed Algorithms **(DFG)** and **(H-DFG)** clearly outperform the classical dual gradient Algorithm **(DG)**. Thus, the practical behaviour observed in simulations certifies the theoretical results derived in the previous sections, where we have proved that the rate of convergence of the proposed algorithms improves

the well known rate of convergence of order $\mathcal{O}(\frac{1}{k})$ for the Algorithm **(DG)**. This behaviour is also valid for the centralized case. Another important aspect consists in the fact that for all algorithms, when the dimension of the problem increases, the distributed version becomes more efficient than the centralized one. This is a consequence of the fact that when the number of busses increases, the level of sparsity of the matrices A and C , characterized in terms of the indices sets \mathcal{S}_i , $\bar{\mathcal{S}}_i$ and \mathcal{N}_i , is high and therefore the Lipschitz constants L_{d_i} are small in comparison with the overall Lipschitz constant L_d (see Section 6 for a more detailed discussion). These differences between L_{d_i} and L_d lead to a grater step size in the case of distributed algorithms in comparison with the centralized ones, thus the distributed algorithms perform faster.

Further, we are also interested in analyzing the behaviour of the proposed algorithms in terms of the primal suboptimality and feasibility violation. For this purpose we consider the 39 bus New England system (pws₄). For this test case we have a number of $M = 39$ buses, $M_g = 10$ generator units and $\bar{M} = 46$ lines between buses. We let the Algorithms **(DFG)** and **(H-DFG)** perform a number of 4000 iterations and we show in Figure 1 the evolution of primal suboptimality and feasibility violation for each algorithm.

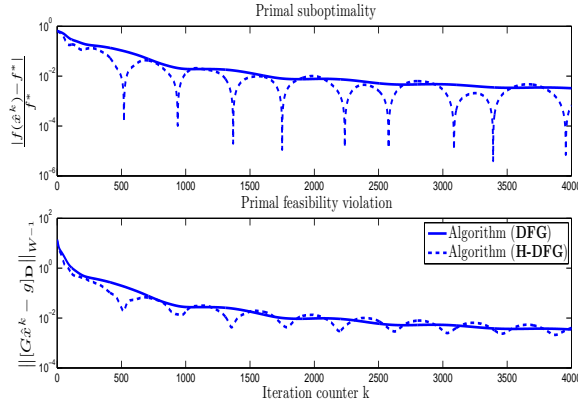


Fig. 1 Comparison between Algorithms **(DFG)** and **(H-DFG)**.

We can observe that, on the one hand, the Algorithm **(H-DFG)** is faster than **(DFG)** but on the other hand both primal suboptimality and primal feasibility for Algorithm **(H-DFG)** have an oscillating behaviour, while in the case of Algorithm **(DFG)** these quantities have a smooth evolution.

For Algorithm **(DFG)** we also plot in Figure 2 the real number of iterations observed in practice and the theoretic number of iterations derived in Section 3. We can observe from Figure 2 that the estimates obtained for the number of iterations are closed to the real number of iterations performed by the algorithm in practice.

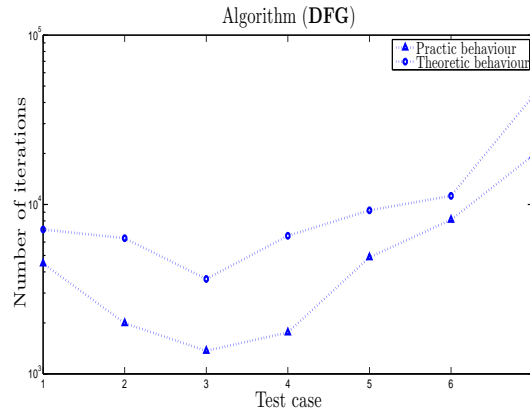


Fig. 2 Comparison between the real number of iterations performed by Algorithm (DFG) in practice and the theoretic number of iterations.

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